Second Order Logic

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SOL is a very expressive formal language which differs from *FOL* in the following respects:

- Syntax: Quantified set and relation variables

$$\begin{array}{ll} \forall x \varphi & \text{for all individuals } \varphi \text{ holds} \\ \forall X \varphi & \text{for all properties, } \varphi \text{ holds} \\ \forall X^2 \varphi & \text{for all binary relations, } \varphi \text{ holds} \end{array}$$

- Semantics: So we have to give meaning to the new variables.
  - Standard: A set variable ranges over the power set of the individual's universe and a relation variable ranges over the power set of the Cartesian product of the individual's universe.
  - Non-standard: Both sets and relation variables range over a subset of the power set of the individual's universe
- Overwhelming expressive power of second order logic with standar semantics. We pay a high price, we sacrifice the logic.
  - compactness fails
  - Löwenheim-Skolem fails.
  - completeness fails

Our λ-language $(\lambda - L_2)$ will have:

1. Connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$
2. Quantifiers: $\forall$, $\exists$
3. Abstractor: λ
4. Parentheses: *),(*.
5. Equality symbols: E, E_1, E_2, ... (for individuals and relations).
6. Falsity: $\perp$
7. Variables:
    1. A set $\mathcal{V}$ of individual variables: x, y, z, etc..
    2. For each $n \in \mathbb{N}$, a set $\mathcal{V}_n$ of $n - ary$ relation variables: X^n, Y^n, Z^n, etc.

1. We will consider a countable infinite set, **OPER.CONS**, including:
    1. Zero-ary function constants: a, b, c, etc
    2. For each $n \in \mathbb{N}$, a set of n-ary function constants: f^n, g^n, h^n, etc.
    3. For each $n \in \mathbb{N}$, a set of n-ary relation constants: R^n, S^n, T^n, etc.

Terms:

$$x \mid b \mid f\tau_1 \ldots \tau_n$$

Predicates:

$$X^n \mid P^n \mid E \mid E_1 \mid E_2 \mid \ldots \mid \lambda x_1 \ldots x_n \psi$$

we can write $t = \tau$ instead of $Et\tau$

Formulas:

Atoms

$$\Pi\tau_1 \ldots \tau_n \mid E\tau_1\tau_2 \mid \ldots \mid E_n \Pi^n \Psi^n \mid$$

Boolean

$$\neg\varphi \mid (\varphi \vee \psi) \mid (\varphi \wedge \psi) \mid (\varphi \rightarrow \psi) \mid (\varphi \leftrightarrow \psi)$$

Quantifiers

$$\forall x\varphi \mid \exists x\varphi \mid \forall X^n\varphi \mid \exists X^n\varphi \mid$$

Comprehension schema, Z^n not free at φ:

$$\exists Z^n \forall x_1 \ldots x_n (Z^n x_1 \ldots x_n \leftrightarrow \varphi)$$

Some instances of the schema:

Example

1. $\exists Z \forall x (Zx \leftrightarrow \neg Rx)$
2. $\exists Z \forall x (Zx \leftrightarrow (Rx \wedge \neg Sx))$
3. $\exists Z \forall x (Zx \leftrightarrow \exists y\ Rxy)$
4. $\exists Z \forall x (Zx \leftrightarrow x = x)$
5. $\exists Z^2 \forall x \forall y (Z^2 xy \leftrightarrow R^2 yx)$
6. The restriction on the variable Z^n cannot be harmlessly dropped. Otherwise we would have monsters like this formula $\exists Z \forall x (Zx \leftrightarrow \neg Zx)$.

$$\mathfrak{A} = \left\langle \mathbf{A}, \langle \mathbf{A}_n \rangle_{n \geq 1}, \left\langle C^{\mathfrak{A}} \right\rangle_{C \in OPER.CONS} \right\rangle$$

1. $\mathbf{A} \neq \varnothing$, the universe of individuals is a non-empty set
2. $\mathbf{A}_n = \wp \mathbf{A}^n$, the universe of $n - ary$ relations is the power set of the $n - ary$ cartesian product of $\mathbf{A}$.
   - The $n - ary$ relational universe contains all possible $n - ary$ relations on $\mathbf{A}$
   - IMPORTANT: the concept of a subset is not defined in the logic, it is just taken from the metatheory of sets as a "logical concept"
3. $R^{\mathfrak{A}} \subseteq \mathbf{A} \times \ldots \times \mathbf{A}$; $R^{\mathfrak{A}}$ is an $n - ary$ relation on individuals.
4. $f^{\mathfrak{A}} : \mathbf{A} \times \ldots \times \mathbf{A} \longrightarrow \mathbf{A}$; $f^{\mathfrak{A}}$ is an $n - ary$ function on individuals. For individual constants the condition reduces to belonging to $\mathbf{A}$

- As usual: Satisfiability, validity and consequence $\Sigma \models_{\mathfrak{S}\mathfrak{S}} \varphi$

Here are a few but significant examples of the expressive power of *SOL*

1. **Arithmetical induction**

$$\forall X(Xc \wedge \forall x(Xx \to X\sigma x) \to \forall xXx)$$

All properties shared by zero and the successor of each number having the property is a property of all numbers.

2. **Finiteness**

$$\alpha := \forall F(\forall xy(Fx = Fy \to x = y) \to \forall x \exists yx = Fy)$$

Every one-to-one function on the universe is also onto

3. **Infinity**

$$\varphi_\infty := \neg\alpha$$

4. **Well ordering**

$$\forall X(\exists yXy \to \exists u(Xu \wedge \forall z(Xz \to u \leq z)))$$

where $\leq$ is an ordering

1. **Countable** and **uncountable**. We can formalize this: "there is a strict linear ordering on the universe such that every element has only finitely many predecessors".
2. **Generalized continuum hypothesis**; that is, between the cardinality of an infinite set and the cardinality of it power set there aren't intermediate cardinalities.

$$\forall XY(\inf(X) \wedge Y \backsim \wp X \rightarrow \forall Z(Z \subseteq Y \rightarrow Z \preceq X \vee Z \backsim Y))$$

for every infinite set X and every Y such that Y can be coordinated with the power set of X,then every subset of Y is either coordinable with Y or has less power than X.

3. **Most R are S**

$$\begin{aligned} \neg\exists X^2(\forall x(\exists y X^2 xy &\leftrightarrow Rx \wedge Sx) \wedge \forall x(\exists y X^2 yx \rightarrow Rx \wedge \neg Sx) \\ \wedge\forall xyz(X^2 xy \wedge X^2 xz &\rightarrow y = z) \wedge \forall xyz(X^2 xy \wedge X^2 zy \rightarrow x = z) \end{aligned}$$

There is not inyective function from $R \cap S$ to $R - S$

Two questions:

- Why are we interested in validities?
- Can we generate them mechanically?

Possible answers:

- Validities are describing the logic in itseld, the laws of mathematical thought and reasoning
- We can try
- Motivations for having a calculus
  1. easier to implement than semantics
  2. better model of reasoning process
  3. limited mathematical resources
  4. psycological advantages
  5. easier to check correctness of proofs

(HI) Hypothesis introduction

$$\overline{\Omega \hookrightarrow \varphi}$$

when $\varphi \in \Omega$

(M) Monotony

$$\frac{\Omega \hookrightarrow \varphi}{\Theta \hookrightarrow \varphi}$$

when $\Omega \subseteq \Theta$.

(PC) Proof by cases **(PC)** and Non contradiction **(NC)**

$$\frac{\begin{array}{l}\Omega \quad \psi \hookrightarrow \varphi \\ \Omega \ \neg\psi \hookrightarrow \varphi\end{array}}{\Omega \qquad \hookrightarrow \varphi} \quad \frac{\begin{array}{l}\Omega \ \neg\varphi \hookrightarrow \psi \\ \Omega \ \neg\varphi \hookrightarrow \neg\psi\end{array}}{\Omega \qquad \hookrightarrow \varphi}$$

(IDA) Introducing disjunction in the antecedent

$$\frac{\begin{array}{lll}\Omega & \varphi & \hookrightarrow \gamma \\ \Omega & \psi & \hookrightarrow \gamma\end{array}}{\Omega\ \varphi \vee \psi \hookrightarrow \gamma}$$

(IDC) Introducing disjunction in the consequent

$$\frac{\Omega \hookrightarrow \varphi}{\Omega \hookrightarrow \varphi \vee \psi} \quad \textit{and} \quad \frac{\Omega \hookrightarrow \varphi}{\Omega \hookrightarrow \psi \vee \varphi}$$

(IPA) Introducing individual particularization in the antecedent

$$\frac{\Omega\ \ \varphi\frac{y}{x} \hookrightarrow \psi}{\Omega\ \ \exists x\varphi \hookrightarrow \psi}$$

$y \notin \mathrm{FREE}(\Omega \cup \{\exists x\varphi,\ \psi\})$, x and y are individual variables.

(IPC) Introducing individual particularization in the consequent

$$\frac{\Omega \hookrightarrow \varphi\frac{\tau}{x}}{\Omega \hookrightarrow \exists x\varphi}$$

(RE) Reflexivity of equality for individuals

$$\overline{\hookrightarrow \tau = \tau}$$

(ES) Equals substitution for individuals $\dfrac{\Omega \qquad \hookrightarrow \varphi\frac{\tau}{x}}{\Omega\ \tau = t \hookrightarrow \varphi\frac{t}{x}}$

(Ext.) Extensionality

$$\overline{\hookrightarrow \forall X^n Y^n (X^n = Y^n \longleftrightarrow \forall x_1 \ldots x_n (X^n x_1 \ldots x_n \longleftrightarrow Y^n x_1 \ldots x_n))}$$

(IPA)^n Introducing relation quantification in the antecedent

$$\frac{\Omega \quad \varphi\frac{Y^n}{X^n} \hookrightarrow \psi}{\Omega \quad \exists X^n\varphi \hookrightarrow \psi}$$

(where X^n and Y^n are n-ary relation variables
$Y^n \notin \mathrm{FREE}(\Omega \cup \{\exists X^n\varphi, \psi\})$
(I.P.C)^n Introducing Relation Quantification in the Consequence

$$\frac{\Omega \hookrightarrow \varphi\frac{\Pi^n}{X^n}}{\Omega \hookrightarrow \exists X^n\varphi}$$

(where Π^n is an n-ary predicate and X^n an n-ary relation variable)

(IAC) Introducing abstraction in the consequent

$$\frac{\Omega \hookrightarrow \varphi\frac{\tau_1 \ldots \tau_n}{x_1 \ldots x_n}}{\Omega \hookrightarrow \lambda x_1 \ldots x_n \varphi \tau_1 \ldots \tau_n}$$

($x_1, \ldots, x_n$ are pairwise distinct individual variables)
(IAA) Introducing abstraction in the antecedent

$$\frac{\Omega \quad \varphi\frac{\tau_1 \ldots \tau_n}{x_1 \ldots x_n} \hookrightarrow \psi}{\Omega \quad \lambda x_1 \ldots x_n \varphi \tau_1 \ldots \tau_n \hookrightarrow \psi}$$

($x_1, \ldots, x_n$ are pairwise distinct individual variables)
(CS) Comprehension schema

$$\frac{}{\hookrightarrow \exists X^n \forall x_1 \ldots x_n (X^n x_1 \ldots x_n \longleftrightarrow \varphi)}$$

where $X^n \notin \mathrm{FREE}(\varphi)$

Desired metaproperties of a calculus, present in FOL

- Soundness: If $\vdash \varphi$ then $\models \varphi$
- Completeness
  - Strong: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$
  - Weak: If $\models \varphi$ then $\vdash \varphi$
- Compactness

  If $\Sigma \models \varphi$ then there is a finite $\Delta \subseteq \Sigma$ such that $\Delta \models \varphi$

- Löwenheim-Skolem

  Δ has an infinite model implies Δ has a countable model

Counterpart of the great expressive power

We get the following results:

- PA^2 is categorical.

$$\mathfrak{A}, \mathfrak{B} \in Mod(\Pi) \Rightarrow \mathfrak{A} \cong \mathfrak{B}$$

  Any two models of Π, Peano arithmetic, are isomorphic. This is so because second order formulation of induction is strong enough as to prevent the formation of $Z-$chains.

- Compactness fails

  There is $\Sigma \models \varphi$ s.t. for no finite $\Delta \subseteq \Sigma$ we have $\Delta \models \varphi$

  The reason is that we can express infinity.

- Löwenheim-Skolem fails.

  Δ has an infinite model but not a countable model

  Think of the formula expressing that the universe is uncountable. This formula can't have countable models.

- Strong completeness fails.

$$\text{There are } \Sigma, \varphi \text{ s.t } \Sigma \models \varphi \text{ but } \Sigma \nvdash \varphi$$

  (In the strong sense completeness fails because compactness does)

- Weak completeness fails

$$\text{There is } \varphi \text{ s.t. } \models \varphi \text{ but } \nvdash \varphi$$

1. This follows from:
   - Arithmetic can be characterized up to isomorphism using a finite axiomatization
   - Gödel formula being true in the structure of natural numbers but not provable from PA^2.
2. There is another proof of the incompleteness of second order logic which is also based upon the expressive power of *SOL*,using the fact that the generalized continuum hypothesis is expressible in second order logic. The proof was written by Ildiko Sain, Agness Kuruzc and myself.

Standard Structures

- SOL with standard semantics has:
  1. an extraordinary expressive power
  2. poor logical properties
- Great expressive power
  1. PEANO AXIOMS are categorical
- Effects of expressive power (the other side of the coin)
  1. COMPACTNESS fails
  2. LÖWENHEIM-SKOLEM fails
  3. COMPLETENESS also fails
- A LOGIC IS LIKE A SCALE
- To retain logical properties we introduce non-standard semantics

1. Prove that this formula is a theorem

$$\exists X x\ Xx$$

2. Prove the uncompactness of SOL. Let

$$\Phi = \{\varphi_n \mid n \geq 2\}$$

where

$$\varphi_n := \exists x_1 \ldots x_n \bigwedge_{i \neq j} x_i \neq x_j$$

Use the formula

$$\varphi_\infty$$