Second Order Logic with Henkin's Semantics

María Manzano mara@usal.es

USAL

January 2012

Let Π be the most famous set of axioms in the world; namely,

- $1P := \forall x \neg c = \sigma x$
- $2P := \forall xy(\sigma x = \sigma y \rightarrow x = y)$
- $3P := \forall X(Xc \wedge \forall z(Xz \rightarrow X\sigma z) \rightarrow \forall x\, Xx)$

We will say that a model of these axioms is a *Peano model*.
Let

$$PA^2 = CON(\Pi) = \{\varphi \in SENT(L_2) \mid \Pi \models \varphi\}$$

- **Theorem:** Any two Peano models are isomorphic
- **Corollary:** $PA^2 = \mathrm{THEO}(\mathcal{N})$

- **Theorem: Incompleteness of Peano Arithmetic**

There is a sentence γ such that $\gamma \in \text{THEO}(\mathcal{N})$ but it is not the case that $PA^2 \vdash_{CAL} \gamma$ in any second order calculus.

- **Note:** The formula γ is the well known Gödel formula asserting of itself that it is not a theorem of Peano arithmetic.
  1. It is in fact not a theorem of PA^2
  2. It holds in the standard model of natural numbers, $\mathcal{N}$.

- **Theorem: Incompleteness of SOL**

There is a sentence φ such that $\models_{\mathcal{S}.\mathcal{S}} \varphi$ but it is not the case that $\vdash_{CAL} \varphi$.

- **Proof**

  1. By Gödel's theorem stated above, $\gamma \in \mathrm{THEO}(\mathcal{N})$.

- **Theorem: Incompleteness of SOL**

There is a sentence φ such that $\models_{\mathcal{S}.\mathcal{S}} \varphi$ but it is not the case that $\vdash_{CAL} \varphi$.

- **Proof**

1. By Gödel's theorem stated above, $\gamma \in \mathrm{THEO}(\mathcal{N})$.
2. So, $\gamma \in PA^2$ —since $\mathrm{THEO}(\mathcal{N}) = PA^2$ $=\mathrm{CON}(\Pi)-$

- **Theorem: Incompleteness of SOL**

There is a sentence φ such that $\models_{\mathcal{S}.\mathcal{S}} \varphi$ but it is not the case that $\vdash_{CAL} \varphi$.

- **Proof**

1. By Gödel's theorem stated above, $\gamma \in \mathrm{THEO}(\mathcal{N})$.
2. So, $\gamma \in PA^2$ —since $\mathrm{THEO}(\mathcal{N}) = PA^2 = \mathrm{CON}(\Pi)$–
3. Take $\varphi \equiv 1P \wedge 2P \wedge 3P \rightarrow \gamma$

- **Theorem: Incompleteness of SOL**

There is a sentence φ such that $\models_{\mathcal{S}.\mathcal{S}} \varphi$ but it is not the case that $\vdash_{CAL} \varphi$.

- **Proof**

1. By Gödel's theorem stated above, $\gamma \in$ THEO$(\mathcal{N})$.
2. So, $\gamma \in PA^2$ —since THEO$(\mathcal{N}) = PA^2$ =CON(Π)–
3. Take $\varphi \equiv 1P \wedge 2P \wedge 3P \rightarrow \gamma$
4. Thus, $\models_{\mathcal{S}.\mathcal{S}} \varphi$ but not $\vdash_{CAL} \varphi$ —otherwise we will have $PA^2 \vdash_{CAL} \gamma$ contradicting Gödel's Theorem stated above—

- What is the conclusion of all this?

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:
  - expresive power

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:
    - expresive power
    - computability power

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:
  - expresive power
  - computability power
- You cannot have both

Standard Semantics

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:
  - expresive power
  - computability power
- You cannot have both
- To retain logical properties we introduce non-standard semantics

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:
  - expresive power
  - computability power
- You cannot have both
- To retain logical properties we introduce non-standard semantics
- We can admit that we had made several mistakes when defining the semantics of *SOL* ; the metatheory of *ZFC* was tied to our object language. This produces a "side effect" which brings about theorems which are consequences of the way of modelling the phenomenon under investigation and, therefore, do not illuminate the basic nature of the phenomenon (higher order reasoning) itself.

- What is the conclusion of all this?
- A LOGIC IS LIKE A SCALE:
  - expresive power
  - computability power
- You cannot have both
- To retain logical properties we introduce non-standard semantics
- We can admit that we had made several mistakes when defining the semantics of *SOL* ; the metatheory of *ZFC* was tied to our object language. This produces a "side effect" which brings about theorems which are consequences of the way of modelling the phenomenon under investigation and, therefore, do not illuminate the basic nature of the phenomenon (higher order reasoning) itself.
- Henkin gave an absolute version of second order logic

- How did Henkin prove, in 1949, the completeness theorem for type theory?

Non-Standard Semantics

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic
  1. **Standard structures:** notion of subset is a “logical" concept,.taken from the background set theory

Non-Standard Semantics

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic
  1. **Standard structures:** notion of subset is a "logical" concept,.taken from the background set theory
  2. **Non-standard structures:** The notion of subset is explicitly given within each model. So it is treated as a concept "defined in the structure".

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic
  1. **Standard structures:** notion of subset is a “logical" concept,.taken from the background set theory
  2. **Non-standard structures:** The notion of subset is explicitly given within each model. So it is treated as a concept “defined in the structure".
- Roughly posed, the idea is very simple: the set of validities is so wide because our class of standard structures is too small.

Non-Standard Semantics

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic
  1. **Standard structures:** notion of subset is a “logical" concept,.taken from the background set theory
  2. **Non-standard structures:** The notion of subset is explicitly given within each model. So it is treated as a concept “defined in the structure".
- Roughly posed, the idea is very simple: the set of validities is so wide because our class of standard structures is too small.
- We are very restricted when asking the relational universes of any model to contain all possible relations (where possible means in the background set theory used as metalanguage).

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic
  1. **Standard structures:** notion of subset is a "logical" concept,.taken from the background set theory
  2. **Non-standard structures:** The notion of subset is explicitly given within each model. So it is treated as a concept "defined in the structure".
- Roughly posed, the idea is very simple: the set of validities is so wide because our class of standard structures is too small.
- We are very restricted when asking the relational universes of any model to contain all possible relations (where possible means in the background set theory used as metalanguage).
  1. If we also allow non-standard structures, then the set of validities in the very large class of structures which includes both classes is considerably reduced.

- How did Henkin prove, in 1949, the completeness theorem for type theory?
- by changing the semantics and hence the logic
  1. **Standard structures:** notion of subset is a “logical" concept,.taken from the background set theory
  2. **Non-standard structures:** The notion of subset is explicitly given within each model. So it is treated as a concept “defined in the structure".
- Roughly posed, the idea is very simple: the set of validities is so wide because our class of standard structures is too small.
- We are very restricted when asking the relational universes of any model to contain all possible relations (where possible means in the background set theory used as metalanguage).
  1. If we also allow non-standard structures, then the set of validities in the very large class of structures which includes both classes is considerably reduced.
  2. In fact, it coincides with the set of sentences derivable in a calculus which is a simple extension of the first order one.

- Two different logics (the set of validities differs)
- What kind of structures do we want to talk about?
  1. Standard view: the same than in FOL, just one universe. Quantification over sets in SOL is just a device to gain expressiveness
  2. Non-standard view: heterogeneous structures. In standard interpretation we do not have enough structures.
- The concept of a subset
  1. Standard: taken from set theory
  2. Non-standard: is explicitly given
- *The litlle mermaid*

$$\mathfrak{A} = \left\langle \mathbf{A}, \langle \mathbf{A}_n \rangle_{n \geq 1}, \left\langle C^{\mathfrak{A}} \right\rangle_{C \in OPER.CONS} \right\rangle$$

1. $\mathbf{A} \neq \emptyset$, the universe of individuals is a non-empty set.
2. $\emptyset \neq \mathbf{A}_n \subseteq \wp \mathbf{A}^n$ that is, the universe of $n-ary$ relations is a subset of the power set of the $n-ary$ cartesian product of $\mathbf{A}$.
3. $R^{\mathfrak{A}} \subseteq \mathbf{A} \times \ldots \times \mathbf{A}$. Now we add the condition that $R^{\mathfrak{A}} \in \mathbf{A}_n$; every distinguished relation must be a member of the corresponding universe.
4. $f^{\mathfrak{A}} : \mathbf{A} \times \ldots \times \mathbf{A} \longrightarrow \mathbf{A}$; $f^{\mathfrak{A}}$ is an $n-ary$ function on individuals. Now we add the condition that $f^{\mathfrak{A}} \in \mathbf{A}_{n+1}$

- In the two previous items we have added the conditions to ensure that the relations and functions distinguished in the structure are included in the corresponding universes for quantification purposes.

$$\mathfrak{A} = \left\langle \mathbf{A}, \langle \mathbf{A}_n \rangle_{n \geq 1}, \left\langle C^{\mathfrak{A}} \right\rangle_{C \in OPER.CONS} \right\rangle$$

1. $\mathbf{A} \neq \emptyset$, the universe of individuals is a non-empty set.
2. $\emptyset \neq \mathbf{A}_n \subseteq \wp \mathbf{A}^n$ that is, the universe of $n - ary$ relations is a subset of the power set of the $n - ary$ cartesian product of $\mathbf{A}$.
3. $R^{\mathfrak{A}} \subseteq \mathbf{A} \times \ldots \times \mathbf{A}$. Now we add the condition that $R^{\mathfrak{A}} \in \mathbf{A}_n$; every distinguished relation must be a member of the corresponding universe.
4. $f^{\mathfrak{A}} : \mathbf{A} \times \ldots \times \mathbf{A} \longrightarrow \mathbf{A}$; $f^{\mathfrak{A}}$ is an $n - ary$ function on individuals. Now we add the condition that $f^{\mathfrak{A}} \in \mathbf{A}_{n+1}$

- In the two previous items we have added the conditions to ensure that the relations and functions distinguished in the structure are included in the corresponding universes for quantification purposes.
- Satisfiability, validity and consequence as usual $\Sigma \models_{\mathfrak{F}} \varphi$

- Frame consequence implies consequence in standard structures. Since
$$\mathcal{SS} \subseteq \mathcal{F}$$
it is easy to see that for all Γ and φ.
$$\Gamma \models_{\mathcal{F}} \varphi \;\text{ implies }\; \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi,$$

- Frame consequence implies consequence in standard structures. Since

$$\mathcal{S}\mathcal{S} \subseteq \mathcal{F}$$

it is easy to see that for all Γ and φ.

$$\Gamma \models_{\mathcal{F}} \varphi \ \text{ implies } \ \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi,$$

- Frame-satisfiability does not imply satisfiability (in standard structures)

$$\exists xy(\forall X(Xx \leftrightarrow Xy) \wedge \neg x = y)$$

which is frame-satisfiable but not satisfiable in $\mathcal{S}.\mathcal{S}$

- Frame consequence implies consequence in standard structures. Since
$$\mathcal{SS} \subseteq \mathcal{F}$$
it is easy to see that for all Γ and φ.
$$\Gamma \models_{\mathcal{F}} \varphi \text{ implies } \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi,$$
- Frame-satisfiability does not imply satisfiability (in standard structures)
$$\exists xy(\forall X(Xx \leftrightarrow Xy) \wedge \neg x = y)$$
which is frame-satisfiable but not satisfiable in $\mathcal{S}.\mathcal{S}$
- Unsoundness of CAL_2 in frames. The comprehension axiom has occurrences which are not valid in the class of all frames. For instance
$$\exists X \forall x(Xx \leftrightarrow Rx \vee Tx)$$

- Frame consequence implies consequence in standard structures. Since
$$\mathcal{S}\mathcal{S} \subseteq \mathcal{F}$$
it is easy to see that for all Γ and φ.
$$\Gamma \models_{\mathcal{F}} \varphi \text{ implies } \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi,$$
- Frame-satisfiability does not imply satisfiability (in standard structures)
$$\exists xy(\forall X(Xx \leftrightarrow Xy) \wedge \neg x = y)$$
which is frame-satisfiable but not satisfiable in $\mathcal{S}.\mathcal{S}$
- Unsoundness of CAL_2 in frames. The comprehension axiom has occurrences which are not valid in the class of all frames. For instance
$$\exists X \forall x(Xx \leftrightarrow Rx \vee Tx)$$
- We need to guarantee that the universes are closed under

- Frames closed under definability
- The class of general structures is between the smaller class $\mathcal{S}.\mathcal{S}$ and the bigger one $\mathcal{F}$

$$\mathcal{S}.\mathcal{S} \subseteq \mathcal{G}.\mathcal{S} \subseteq \mathcal{F}$$

- For all Γ and φ

$$\Gamma \models_{\mathcal{F}} \varphi \text{ implies } \Gamma \models_{\mathcal{G}.\mathcal{S}} \varphi \text{ implies } \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi$$

- Frames closed under definability
- The class of general structures is between the smaller class $\mathcal{S}.\mathcal{S}$ and the bigger one $\mathcal{F}$

$$\mathcal{S}.\mathcal{S} \subseteq \mathcal{G}.\mathcal{S} \subseteq \mathcal{F}$$

- For all Γ and φ

$$\Gamma \models_{\mathcal{F}} \varphi \text{ implies } \Gamma \models_{\mathcal{G}.\mathcal{S}} \varphi \text{ implies } \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi$$

- We get soundness

$$\Gamma \vdash_{CAL_2} \varphi \text{ implies } \Gamma \models_{\mathcal{G}.\mathcal{S}} \varphi$$

for all Γ and φ

- Frames closed under definability
- The class of general structures is between the smaller class $\mathcal{S}.\mathcal{S}$ and the bigger one $\mathcal{F}$

$$\mathcal{S}.\mathcal{S} \subseteq \mathcal{G}.\mathcal{S} \subseteq \mathcal{F}$$

- For all Γ and φ

$$\Gamma \models_{\mathcal{F}} \varphi \text{ implies } \Gamma \models_{\mathcal{G}.\mathcal{S}} \varphi \text{ implies } \Gamma \models_{\mathcal{S}.\mathcal{S}} \varphi$$

- We get soundness

$$\Gamma \vdash_{CAL_2} \varphi \text{ implies } \Gamma \models_{\mathcal{G}.\mathcal{S}} \varphi$$

for all Γ and φ

- And completeness

$$\Gamma \models_{\mathcal{G}.\mathcal{S}} \varphi \text{ implies } \Gamma \vdash_{CAL_2} \varphi$$

Conclusion

When defining the standard semantics it was tied to ZFC Zermelo Fraenkel set theory and the higher order reasoning was hidden. We change to non-standard semantics and obtain completeness.

- You can not have both
  1. expressive power
  2. good logical properties
- Little mermaid
  1. she got two beautiful legs
  2. she lost he voice