Many-sorted Logic

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Abstract

Translations into many-sorted logic can be seen as the path to completeness in three possible stages: abstract completeness at the first stage, strong completeness at the third. The case of second-order logic can act as the source of inspiration for translations of other logical systems. The idea is to obtain many-sorted structures where the Comprehension Schema restricted to formulas which are translations is always true. This idea of restricting CS to obtain new logics is taken from "Banishing the rule of substitution for functional variables", Henkin (1953).

1. Many-sorted logic
2. Translations into MSL
3. Second-order Logic

Many-sorted logic

Examples

In many branches of philosophy, mathematics and *CS* structures are conceived as many sorted.

- Examples:
  - Euclid's first principle
  - Mathematics:
    - Theory of Groups,
    - Rings,
    - etc
  - Philosophy:
    - Type Theory
  - Computation
  - Reasoning about computation
  - etc.

Euclid's first principle

According to Euclid's first principle:

- *A straight line can be drawn joining any two points.*

In two sorted first-order logic one can use $X, Y, \ldots$ as variables for sort l (straight lines) and $x, y, \ldots$ as variables for sort p (points). To formulate Euclid's principle we write:

$$\forall x \forall y (x \neq y \rightarrow \exists X (Joint(Xxy) \wedge \forall Y (Joint(Yxy) \rightarrow X = Y)))$$

In this example *Joint* is a 3-place predicate symbol relating points and straight lines.

Second-order Structures

- A *second-order standard structure* $\mathcal{A} = \langle \mathbf{A}, \langle \wp(\mathbf{A}^n) \rangle_{n \in \mathbb{N}}, \langle f^{\mathcal{A}} \rangle_{f \in OperSym} \rangle$
  - Includes:
    - the domain of individuals, $\mathbf{A}$
    - a family of second order domains of $n-$ary relations on individuals, $\wp(\mathbf{A}^n)$, for each natural number n.
    - relations and functions in $\langle f^{\mathcal{A}} \rangle_{f \in OperSym}$
- A *second-order general structure* $\mathcal{A} = \langle \mathbf{A}, \langle \mathbf{A}_n \rangle_{n \in \mathbb{N}}, \ldots \rangle$
  - As in standard structures,
    - domain of individuals, $\mathbf{A}$
    - domains of $n-$ary relations for each natural number, $\mathbf{A}_n \subseteq \wp(\mathbf{A}^n)$.
  - To be a general structure,
    - the universes $\mathbf{A}_n$ are not arbitrarily chosen,
    - they must satisfy the *Comprehension Schema*.
  - Therefore, the universes are *closed under definability*.

What is the Logic that best fits our needs?

Many-sorted logic

1. By many-sorted logics, *MSL*-logics, we mean a family of logics with:
    1. more than one sort of variables (at the syntactic level)
    2. each of them ranging over a different domain (at the semantic level).
2. Signature
$$\Sigma = \langle Sort, OperSym, Rank \rangle$$
In Euclid's example: $Rank(Joint) = \langle l, p, p, 0 \rangle$
3. *MSL* has a strongly complete deductive calculus.
4. Good metalogical properties: interpolation, interpretability, etc.

Mantra about MSL

MSL is irrelevant

- Easily convertible into one-sorted logic.
  - Syntactical translation: Relativization of quantifiers.

$$Trans(\exists x^i \phi) \quad := \quad \exists x^i (Q^i x^i \wedge Trans(\phi))$$

  - Conversion of many-sorted structures into one-sorted ones: Unification of domains.
- For any many-sorted structure $\mathcal{A}$ of signature Σ we construct its corresponding one-sorted structure $\mathcal{A}^*$
  - Theorem:

$$\mathcal{A} \models \varphi \iff \mathcal{A}^* \models Trans(\varphi).$$

  - Main Theorem: Let $\Gamma \cup \{\varphi\} \subseteq Sent(L)$.

$$\Gamma \models_{MSL} \varphi \text{ if and only if } Trans(\Gamma) \cup \Pi \models_{FOL} Trans(\varphi)$$

- Conclusion: *MSL* is irrelevant

Do we need many-sorted logic at all?

Price paid for the reduccion into one-sorted logic

- *Naturalness.* One of our aims in studying many-sorted logic was to find a logic matching the many-sorted structures we wanted to talk about.
- *Efficiency.* Deductions in a many-sorted calculus are usually shorter. Efficiency is fundamental in automathed theorem provers.
- *Interpolation.* Feferman showed that many-sorted logic exhibits versions of Craig's interpolation theorem that are stronger (and not reductible to) the one-sorted version.
- *Interpretability.* Hook proved that a many-sorted theory could be interpretable in another many-sorted theory without the corresponding translations being interpretable in one another.

Translations

Currently, the proliferation of logical systems used in mathematics, computer science, philosophy and linguistics makes their relationships between and their possible translations into one another a pressing issue.

- *How can we compare logics?, how can we transfer metaproperties from one logic to another?*
- Answers related to different translations methodologies.
  - From a *proof-theoretical* they emerge the "labelled deductive systems" of Dov Gabbay.
  - From a *model-theoretical perspective*, the "correspondence theory" of Johan van Benthem.
  - From a *suprastructural* point of view, we should highlight the "general logics" of Pepe Meseguer.
- In the three cases you ended up in classical logic.

The paradigm of translation

First order logic

- The "target logic" into which other logics are translated uses to be unsorted first-order classical logic.
- Why should we translate a logic *XL* into *first-order* (unsorted) rather than into *many-sorted logic*?
- General arguments for the convenience of first-order logic are:
  - Once a logic *XL* is translated into first-order logic we don't need but a unique deductive calculus, which is known to be strongly complete, and a unique theorem prover.
  - Further, it is possible to borrow some metaproperties of first-order logic in order to apply then to *XL*.
- But these arguments hold for many-sorted logic too.

Translations into MSL (three levels)

Reduction of XL to Many Sorted Logic

- General plan:
  - Signature of XL transformed into signature for MSL^{XL}
  - Expressions of XL translated into MSL^{XL}
  - Structures of XL converted into many-sorted structures.
- Translation function, *Trans* to translate the formulas

$$Trans\colon \quad Exp(XL) \longrightarrow Exp(MSL^{XL})$$

- Direct conversion of structures, $Conv_1$:

$$Conv_1 \quad St(XL) \longrightarrow St(MSL^{XL})$$

  (Add new universes for definable sets and relations)
- *In the following sections I will present translations into many-sorted logic as the path to completeness in three stages.*

Translating into MSL (three levels)

Level one: representation theorems

Our first goal is to state and prove:

- For every sentence φ of the logic XL,

$$\models_{Str(XL)} \varphi \text{ in } XL \text{ iff } \models_{\mathfrak{S}^*} \forall Trans(\varphi) \text{ in } MSL,$$

  where $\mathfrak{S}^* = Conv_1 Str(XL)$

- The next step. is to replace the semantical restriction to $\mathfrak{S}^*$ for a suitable set $\Delta \subseteq Sen(L^*)$ such that the representation theorem holds; namely, a theorem in the following fashion:
- *Representation Theorem*. There is a recursive set Δ of L^*-sentences with $\mathcal{S}^* \subseteq Mod(\Delta)$ such that, for every L-sentence φ of the logic XL

$$\models_{Str(XL)} \varphi \text{ in } XL \text{ iff } \Delta \models_{Str(MSL^{XL})} \forall Trans(\varphi) \text{ in } MSL$$

Translating into MSL (three levels)

Level one: corollaries

- From Representation theorem it follows *Enumerability* for the logic *XL*.
- Namely, the set of validities of *XL* is recursively enumerable.
- Therefore, *XL* is *complete in an abstract sense*.
- **Remark:** So, we learn that a calculus for *XL* is a natural demand, but we also learn that in *MSL* we can simulate such a calculus and then we could use a theorem prover for *MSL*.

Translating into MSL (three levels)

Level two: the main theorem

Is the representation theorem our top (maximum) goal?

- When the *XL* logic under scrutiny has a concept of logical consequence, we may try to prove the *Main theorem*
  - Reverse conversion of structures, $Conv_2$, should be used.
  - Our goal in defining it is to prove:
- **Lemma 2:** For any $\varphi \in Sent(XL)$ and $\mathcal{B} \in Mod(\Delta)$

$$Conv_2(\mathcal{B}) \models \varphi \quad \text{iff} \quad \mathcal{B} \models \forall Trans(\varphi)$$

- **Main Theorem:** There is a recursive set $\Delta \subseteq Sent(L^*)$ with $\mathfrak{S}^* \subseteq Mod(\Delta)$, such that

$$\Pi \models_{Str(XL)} \varphi \quad \text{iff} \quad Trans(\Pi) \cup \Delta \models_{Str(MXL^*)} Trans(\varphi)$$

for all $\Pi \cup \{\varphi\} \subseteq Sent(XL)$.

Translating into MSL (three levels)

Level two: the main theorem

From Main Theorem we obtain important metalogical results

- **Corollary:** *Compactness* and *Löwenheim-Skolem* for *XL*.
- **Remark:** We already have *Enumerability*. Now, from *Main Theorem* as well *Compactness* and *Löwenheim-Skolem* for *MSL*, we easily obtain *Compactness* and *Löwenheim-Skolem* for *XL*.
- *Therefore, the logic under investigation could have a strong complete calculus. You can either define it or use the many-sorted one.*

Translating into MSL (three levels)

Level three: deductive correspondence

- When the *XL* logic also comes with a deductive calculus, we can try to use the machinery of correspondence to prove, if possible, *Soundness* and *Completeness* for *XL*.
- In order to prove these theorems, one needs to prove a *Theorem of deductive correspondence* between the *XL* calculus and the many-sorted one, modulo Δ.
- So, the next goal is to prove:
- **Theorem of deductive correspondence:** Let Δ be defined as in the Main Theorem, then:

$$Trans(\Pi) \cup \Delta \vdash_{MSL} Trans(\varphi) \text{ if and only if } \Pi \vdash_{XL} \varphi.$$

- The left arrow is usually very easy because you can add to Δ the required axioms. In some cases it is better to use the canonical model construction.

Translating into MSL (three levels)

Level three: soundness and completeness of XL

- Since we already have the *Main theorem* (*), plus *Completeness* and *Soundness* for *MSL* (**), once we get this *Theorem of deductive correspondence* (***) the picture below gives us *Soundness and Strong Completeness* of *XL*:

$$
\begin{array}{ccccr}
Trans(\Pi) \cup \Delta \models_{Str(MSL)} Trans(\varphi) & \Longleftrightarrow & \Pi \models_{Str(XL)} \varphi & & (*) \\
\Updownarrow & & & & (**) \\
Trans(\Pi) \cup \Delta \vdash_{MSL} Trans(\varphi) & \Longleftrightarrow & \Pi \vdash_{XL} \varphi & & (***)
\end{array}
$$

- **Corollary:** *Soundness and Strong Completeness* of *XL*:

$$\Pi \models_{Str(XL)} \varphi \text{ if and only if } \Pi \vdash_{XL} \varphi$$

Second-order Logic

Standard Structures, General Structures and Frames

- Second-order logic, *SOL*, is a very expressive formal language which differs from *FOL* (one-sorted first-order logic) in the following respects:
  1. alphabet,
  2. standard and non-standard semantics
  3. overwhelming expressive power with standard semantics.
- $\mathfrak{S}.\mathfrak{S}$, standard structures and $Val_{\mathfrak{S}\mathfrak{S}}$ validities
- There is no complete calculus for $Val_{\mathfrak{S}\mathfrak{S}}$
- Henkin 1950: "the incompleteness problem is solvable if '*a wider class of models*' (non-standard models) is allowed".
- Therefore, he introduces:
  1. standard structures, $\mathfrak{S}\mathfrak{S}$
  2. general structures, $\mathfrak{G}\mathfrak{S}$
  3. frames, $\mathfrak{F}$.

Second-order Logic

Standard Structures, General Structures and Frames

- The classes of these structures are ordered by set inclusion

$$\mathfrak{S}\mathfrak{S} \subseteq \mathfrak{G}\mathfrak{S} \subseteq \mathfrak{F}$$

  and, accordingly, the sets of validities in each class obey the reverse order

$$Val_{\mathfrak{F}} \subseteq \; Val_{\mathfrak{G}\mathfrak{S}} \subseteq \; Val_{\mathfrak{S}\mathfrak{S}}$$

- In the *semantics of frames*, both sets and relation variables are allowed to range over universes which are subsets of standard universes.
  - this condition alone doesn't guarantee that we have enough sets and relations to deal with the second order capability.
- In the *general models* all the definable ones are included.
  - This is a rather natural demand and it gives completeness.

Translation of SOL into MSL

Completeness of Second-order Logic

We define a syntactical translation from SOL to MSL^{SOL} as well as two conversions of structures, direct and reverse.

- *Syntactical translation*
    - we retain all the symbols in $OperSym$
    - new membership relation symbols, ϵ_n,
- The *syntactical translation* from SOL to this MSL^{SOL} leaves every formula the same, only exception

$$Trans^{SOL}(X^n x_1 \ldots x_n) := \epsilon_n x_1 \ldots x_n X^n$$

- *Direct conversion of structures*
    - *$Conv_1$, from SOL to MSL^{SOL})*
    - imput $\mathcal{A} \in Str(SOL)$ output $\mathcal{A}^* \in Str(MSL^{SOL})$.
    - Basically the same.
- Only exception: *membership relations* we now add.

Translation of SOL into MSL

Completeness of Second-order Logic

- **Proposition:** For every sentence φ of SOL:

  $\models_{\mathfrak{G}\mathfrak{S}} \varphi$ in SOL if and only if $\models_{\mathfrak{S}^*} Trans^{SOL}(\varphi)$ in MSL^{SOL}

  $(\mathfrak{S}^* = Conv_1(Str(SOL)))$
- *Now we ask, can we axiomatize* $\mathfrak{S}^*$*?* and the answer is *YES*.
  - The structures we want to work with are MSL^{SOL} models of Δ (*Extensionality* and *Comprehension* with *Disjoint universes*)
- *Reverse conversion of structures (from* MSL^{SOL} *to* SOL*),* $Conv_2$
- With all these definitions at hand, we now prove:
- **Proposition:** for every many-sorted structure $\mathcal{C} \in Mod(\Delta)$ there is a $SOL-$structure $Conv_2(\mathcal{C}) \in Str(SOL)$ such that, for all sentences

$$Conv_2(\mathcal{C}) \models \varphi \text{ if and only if } \mathcal{C} \models Trans^{SOL}(\varphi)$$

Translation of SOL into MSL

Completeness of Second-order Logic

- At level two one proves:
  - *Representation theorem:*

$$\models_{\mathfrak{G}.\mathfrak{S}} \varphi \text{ in } SOL \text{ iff } \Delta \models_{MSL} Trans^{SOL}(\varphi) \text{ in } MSL^{SOL}.$$

  - *Main theorem:* $\Pi \models_{\mathfrak{G}.\mathfrak{S}} \varphi$ in SOL if and only if $Trans^{SOL}(\Pi) \cup \Delta \models_{MSL} Trans(\varphi)$ in MSL^{SOL}.
- SOL with general semantics has the following properties:
  - *Enumerability*,
  - *Compactness* and
  - *Löwenheim-Skolem*.
- At level 3 one gets:
- **Strong Completeness and Soundness**:.

$$\Pi \models_{\mathcal{G}.\mathcal{S}} \varphi \text{ iff } \Pi \vdash_{SOL} \varphi \text{ for all } \Pi \cup \{\varphi\} \subseteq Sent(L_2)$$

Henkin's paper, 1953

Banishing the rule of substitution for functional variables

- Henkin proposes the *comprehension axiom* as a way to avoid the rule of substitution.
  - two calculi F^* and F^{**} (basically our *MSL* and the *SOL*)
  - two classes of structures (basically our $\mathfrak{F}$ and $\mathfrak{GS}$)
  - Both calculi are complete
    - The advantage is that the property of being a general structure can be axiomatized.
    - Thus,
$$\models_{\mathfrak{GS}} \varphi \Longleftrightarrow \Delta \models_{\mathfrak{F}} \varphi$$
- To isolate calculi between the *MSL* calculus and *SOL*, by weakening comprehension.
- And it is easy to find a semantics for the logic thus defined (the class of structures is placed between $\mathfrak{F}$ and $\mathfrak{GS}$)
- The new logic will also be complete.

Translations into MSL with a Second-order Appearance

- At the first level:
  - Add new universes to the converted structure $Conv_1(XL)$ (for definable sets and relations in XL)
- *Representation Theorem*.
  - Set Δ of L^*-sentences includes *Comprehension* for translations

$$\models_{Str(XL)} \varphi \text{ in } XL \text{ iff } \Delta \models_{Str(MSL^{XL})} \forall Trans(\varphi) \text{ in } MSL$$

- Example: *Modal Logics*.
  - Given a Kripke structure

$$\mathcal{A} = \langle \mathbf{W}, \mathbf{R}, \langle P^{\mathcal{A}} \rangle_{P \in Atom} \rangle$$

we say that $\mathcal{AG}$ is a general structure built on $\mathcal{A}$ if and only if

$$\mathcal{AG} = \langle \mathbf{W}, \mathbf{W}', \mathbf{R}, \epsilon_1^{\mathcal{A}}, \langle P^{\mathcal{A}} \rangle_{P \in Atom} \rangle$$

where $Def \subseteq \mathbf{W}' \subseteq \wp(\mathbf{W})$ (Def includes definable sets in ML)
  - *Strong Completeness* ML (level 3, at the end of the process)