Partial heterogeneous logic for modal logic

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Abstract

In this article language and semantics of the partial and heterogeneous logic of meaningless are presented. This logic is the suitable one to translate modal predicate logic into. After that a sequent calculus for this logic is defined and proof theoretical results proved.

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Contents

1 Introduction

The aim of this paper is to present the semantics and a sequent calculus for the partial heterogeneous logic of Meaningless, PHLM for short.

PHLM is motivated by predicate modal logic. In [6] is shown that the partiality in modal predicate logic came from the most important question about this logic: what is the truthvalue to assign to formulas which interpretation involves objects that don't exist in the domain of the given world?. The most successful answer has been that of Hughes & Cresswell [5] and its *truth-value gaps*, where no truthvalue (true or false) is assigned to these troubled formulas. But many modal formulas are then out of reach of the semantics. To avoid this other problem, the lack of truthvalue must be a third non-classical truthvalue itself.

Basically, PHLM contains the partial connectives and quantifiers, and the partial semantical concepts of validity and logical consequence suitable to capture the meaningless 'meaning' of the *truth-value gaps* in predicate modal logic. It is a three-valued logic with more than one kind of variables. PHLM is not a classical logic but it has been built looking for the maximum of classicality possible.

PHLM is then the suitable partial heterogeneous logic for "classically talking" about the *truth-values gaps* in predicate modal logic. PHLM is also the partial heterogeneous logic containing the Bočvar's three-valued system of connectives (also called Kleene's weak connectives) [4]. This is why PHLM is called the partial and heterogeneous logic of meaningless.

2 Partial and Heterogeneous Logic of Meaningless PHLM

PHLM language and semantics are presented in the following. All the proofs are skipped and can be founded in [6].

2.1 Partial Connectives and Quantifiers

In this section the partial system of connectives and quantifiers appropriate to a '*meaningless*' semantics is presented. We deal with three truth-values, the classical true (T) and false (F) truth-values and null (N) corresponding to the third value (*meaningless*).

It is known that the suitable connectives (negation, conjunction and disjunction) for this semantics are Bočvar's (see [4]). They preserve the classical connectives and they generate N from N.

The motivation behind Bočvar's three-valued system is to avoid the logical paradoxes. Bočvar declared as propositions without meaning the ones involve in paradoxes, and the *third-value* is '*paradoxical*' or '*meaningless*'. The tables are as follows (Kleene in [7] has similar connectives to these ones, which are named '*weak*'):

α	$\neg\alpha$
T	F
F	T
N	N

		$\alpha \wedge \beta$			$\alpha \vee \beta$			$\alpha \rightarrow \beta$	
	$\beta\{$	T	F	N	T	F	N	T	F	N
α	T	T	F	N	T	T	N	T	F	N
	F	F	F	N	T	F	N	T	T	N
	N	N	N	N	N	N	N	N	N	N

As in classical logic, it holds that $\alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$ and $\alpha \rightarrow \beta \equiv \alpha \vee (\neg\beta)$. So $\{\neg, \vee\}$ generate the other two connectives, but they are not a functionally complete set of connectives, so we need other new connectives.
We will see in the next section that we need new partial connectives.

Given the truthvalues set $A_0 = \{T,\ F,\ N\}$, a *n-ary connective* for A_0 is a function $\Theta : A_0^n \longrightarrow A_0$ $(n \geq 0)$
Given a set C of connectives for A_0, we denote $COMP.C$ the set of functions obtained by composition of functions of C.

A set of connectives C for a truthvalues set A_0 is *functionally complete* if for any n-ary function $\Theta : A_0^n \longrightarrow A_0$ $(n \geq 0)$ there is a n-ary function $\Phi : A_0^n - \rightarrow A_0$ such that $\Phi \in COMP.C$ and $\Theta(\bar{x}) = \Phi(\bar{x})$ for all $\bar{x} \in A_0^n$.

The idea is that any possible connective be expressible by composing the ones of a functional complete set of connectives. Thus in the language of PHLM we only need to define symbols for the connectives in that set.

We consider the new connectives V *(verification)* and ξ *(difalsification)* defined in the following table:

α	$V\alpha$	$\xi\alpha$
T	T	T
F	F	N
N	F	T

And we obtain the following theorem.

1. $\{\neg, \vee, V\}$ *is not functionally complete for* $A_0 = \{T, F, N\}$.
2. $\{\neg, \vee, V, \xi\}$ *is a functionally complete set of connectives for* $A_0 = \{T, F, N\}$.

Now we will define the truth-value tables of the partial quantifiers $\exists$ and $\forall$. In order to conserve the classical quantifiers when restricted to values T and F we define them as follows:

If $\mathcal{I} = \langle \mathcal{A}, \mathcal{H} \rangle$ is an interpretation of first order logic, with $\mathcal{A}$ a first order structure and $\mathcal{H}$ a classical variable assignment, and we note $\mathcal{I}_x^a = \langle \mathcal{A}, \mathcal{H}_x^a \rangle$ where $\mathcal{H}_x^a = (\mathcal{H} - \langle x, \mathcal{H}(x) \rangle) \cup \{\langle x, a \rangle\}$

$$\mathcal{I}(\exists x \varphi) = \begin{cases} T \Leftrightarrow \{a \in A \ / \ \mathcal{I}_x^a(\varphi) = T\} \neq \emptyset \\ F \Leftrightarrow \{a \in A \ / \ \mathcal{I}_x^a(\varphi) = F\} = A \\ N \Leftrightarrow \{a \in A \ / \ \mathcal{I}_x^a(\varphi) = N\} \neq \emptyset \text{ and } \{a \in A \ / \ \mathcal{I}_x^a(\varphi) = T\} = \emptyset \end{cases}$$

$$\mathcal{I}(\forall x \varphi) = \mathcal{I}(\neg \exists x \neg \varphi)$$

2.2 Partial Heterogeneous Language

The definitions of this section are based on [8] with the obliged by partiality innovations.
A *PHLM signature* is a pair $\Sigma = \langle GEN, RANK \rangle$ where:

- GEN (the set of *genera*) is an index set with $0 \in GEN$ (0 is the *genus* corresponding to the *truth-values*, the other indexes or *genera* correspond to an *individual variable genus*, each one).
- $RANK$ is a function from $OPER.SYM$ (set of *operation symbols*) into $S_\omega(GEN) \cup (\omega - \{0\})$ (set of *types*), where ω is the set of natural numbers with the usual order and $S_\omega(GEN) = \bigcup_{n \in \omega} \{GEN^n\}$ is the set of all finite sequences of GEN.
- $[OPER.SYM]_\alpha = \{f \in OPER.SYM \ / \ RANK(f) = \alpha\}$ (set of *operation symbols of type* α) with $\alpha \in S_\omega(GEN) \cup (\omega - \{0\})$ define a partition in $OPER.SYM$.

A *PHLM language* $\mathcal{L}$ of signature $\Sigma = \langle GEN, RANK \rangle$ is a set of formal symbols $ALF(\mathcal{L}) \cup EXPR(\mathcal{L})$, where:

- $ALF(\mathcal{L}) = OPER.SYM \cup \mathcal{V} \cup \{\exists\}$. With $OPER.SYM$ the set of *operation symbols of the language* and $\{\neg, \vee, V, \xi, ,=\} \subseteq OPER.SYM$.
  - ⋆ $\mathcal{V} = \cup \{\mathcal{V}_i \ / \ i \in GEN - \{0\}\}$ is the set of *variables*, and $\mathcal{V}_i$ the set of *variables of genus* i.
  - ⋆ We will consider that the propositional constants $\top$ (*truth-hood*) and $\perp$ (*falsehood*) are in $ALF(\mathcal{L})$ if required.
- $EXPR(\mathcal{L})$, the set of the *expressions of the language* $\mathcal{L}$ is defined inductively:
  1. A variable of genus i is an *expression of genus* i.

2. If $f \in OPER.SYM$ with $RANK(f) = \langle i_0, i_1, ..., i_m \rangle$ and $\varepsilon_1, ..., \varepsilon_m$ are expressions of genus $i_1, ..., i_m$ respectively, then $f\varepsilon_1...\varepsilon_m$ is an *expression of genus* i_0 (if $i_0 \neq 0$ we will use the symbol f as a strict operation symbol, and if $i_0 = 0$ we will use the symbol r as a strict (typed) relation symbol). If $R \in OPER.SYM$ with $RANK(R) = n \in (\omega - \{0\})$ and $\varepsilon_1, ..., \varepsilon_n$ are expressions of any genus different from 0, then $R\varepsilon_1...\varepsilon_n$ is an *expression of genus* 0.
3. If ε is an expression of genus 0 and x_i is a variable of genus i, then $\forall x_i \varepsilon$ and $\exists x_i \varepsilon$ are *expressions of genus* 0.

- We note $TERM(\mathcal{L})$ the set of all expressions of $\mathcal{L}$ of genus different from 0, *terms of the language*, and $TERM_i(\mathcal{L})$ the set of terms of the specific genus $i \neq 0$. And we will note $FORM(\mathcal{L})$ the set of the expressions of $\mathcal{L}$ of genus 0, *formulas of the language* $\mathcal{L}$.

We take as primitive connectives the set $\{\neg, \vee, V, \xi\}$ and as primitive quantifier $\exists$, and we have the usual definitions of $\vee, \wedge, \leftrightarrow$, and $\forall$ for the partial connectives and quantifier:

$\varphi \to \psi \equiv \neg\varphi \vee \psi$
$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$
$\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$
$\forall x_i \varphi \equiv \neg\exists x_i(\neg\varphi)$.

Notation.
$FREE(\varepsilon)$ is the set of all variables that freely occur in the expression ε, that is, variables appearing in ε and not affected by any quantifier
$FREE_i(\varepsilon)$ is the set of free variables of genus i.

If x_i is a variable of genus i and τ is a term, *the substitution of the term* τ *for the variable* x_i in an expression ε, $S^{\tau}_{x_i}(\varepsilon)$, is defined recursively:

1. If z_j is a variable, $S^{\tau}_{x_i}(z_j) \equiv \begin{cases} \tau \text{ if } x_i = z_j \\ z_j \text{ otherwise} \end{cases}$
2. If $f \in OPER.SYM$ and $RANK(f) = \langle i_0, i_1, ..., i_m \rangle$
   - $S^{\tau}_{x_i}(f\tau_1...\tau_m) \equiv \begin{cases} fS^{\tau}_{x_i}(\tau_1)...S^{\tau}_{x_i}(\tau_m) \text{ if } x_i, \tau \text{ have the same genus} \\ f\tau_1...\tau_m \text{ otherwise} \end{cases}$
   - $S^{\tau}_{x_i}(\neg\varphi) \equiv \neg(S^{\tau}_{x_i}(\varphi))$
   - $S^{\tau}_{x_i}(\varphi \vee \psi) \equiv S^{\tau}_{x_i}(\varphi) \vee S^{\tau}_{x_i}(\psi)$
   - $S^{\tau}_{x_i}(V\varphi) \equiv V(S^{\tau}_{x_i}(\varphi))$
   - $S^{\tau}_{x_i}(\xi\varphi) \equiv \xi(S^{\tau}_{x_i}(\varphi))$
   - $S^{\tau}_{x_i}(R\tau_1...\tau_n) \equiv RS^{\tau}_{x_i}(\tau_1)...S^{\tau}_{x_i}(\tau_n)$ if $RANK(R) = n$

3. If $\varphi \in FORM$,

$$S^{\tau}_{x_i}(\exists z_j \varphi) \equiv \begin{cases} \exists z_j \varphi \ \text{ if } x_i \notin FREE(\exists z_j \varphi) \\ \exists z_j S^{\tau}_{x_i}(\varphi) \ \text{ if } x_i \in FREE(\exists z_j \varphi) \ \text{ and } \ z_j \notin FREE(\tau) \\ \exists y_j S^{\tau}_{x_i}(S^{y_j}_{z_i} \varphi) \ \text{ if } x_i \in FREE(\exists z_j \varphi),\ z_j \in FREE(\tau) \text{ and} \\ \qquad\qquad y_j \in \mathcal{V}_j \text{ is the first variable non appearing in } \varphi \end{cases}$$

2.3 Partial Heterogeneous Semantics

A *partial heterogeneous structure* is a pair $\mathcal{A} = \langle \{A_i\}_{i \in GEN}, \{f^{\mathcal{A}}\}_{f \in OPER.SYM} \rangle$

- A_i is the *universe of genus* $i \neq 0$ of $\mathcal{A}$ (for each $i \in GEN$)
  $A_0 = \{V, F, N\}$ is the *universe of truthvalues.*
- $\{f^{\mathcal{A}}\}_{f \in OPER.SYM}$ is a family of *functions* such that:
  - a. $f^{\mathcal{A}} : A_{i_1} \times ... \times A_{in} \longrightarrow A_{i_0}$ $[RANK(f) = (i_0, ..., i_n)]$
    - a.1. if $i_0 \neq 0$ then $f^{\mathcal{A}}$ is an *operation function*, assigning an individual of the universe of genus i_0 to a n-pla of genus $A_{i_1} \times ... \times A_{in}$.
    - a.2. if $i_0 = 0$ then $f^{\mathcal{A}} \equiv r^{\mathcal{A}} : A_{i_1} \times ... \times A_{in} \longrightarrow A_0$ is the characteristic function of a *relation*, assigning a truth-value to a n-pla of $A_{i_1} \times ... \times A_{in}$.
  - b. $f^{\mathcal{A}} \equiv R^{\mathcal{A}} : \left(\bigcup_{i \in GEN - \{0\}} A^n_i \right) \longrightarrow A_0$ $[RANK(f) = n \in \omega - \{0\}]$, is the characteristic function of a *relation* defined in the union of all the universes.
  - c. In particular $\{\neg, \vee, V, \xi, =\} \subseteq OPER.SYM$ with:
    - c.1. $\neg^{\mathcal{A}} : A_0 \longrightarrow A_0$ and $\vee^{\mathcal{A}} : A_0 \times A_0 \longrightarrow A_0$ are *Bočvar‘s negation and disjunction connectives.*
    - c.2. $V^{\mathcal{A}} : A_0 \longrightarrow A_0$ and $\xi^{\mathcal{A}} : A_0 \longrightarrow A_0$ are the *verification and difalsification partial connectives*
    - c.3. $=^{\mathcal{A}}: \left(\bigcup_{i \in GEN - \{0\}} A \right)_i \longrightarrow A_0$, (with $=^{\mathcal{A}} (a, b) = T$ iff a and b are exactly the same individual). The interpretation of the equality symbol is identity.

The *cardinality* of a heterogeneous structure is the sum of the cardinalities of its universes.

Given a language $\mathcal{L}$ and a structure $\mathcal{A}$ with the same signature Σ, we shall show how every term of $\mathcal{L}$ can denote an element of the universes of $\mathcal{A}$ and how we can associate a truth value to every formula of $\mathcal{L}$. To do that we first define the fundamental concepts of the PHLM semantics.

An *assignment* over a structure $\mathcal{A}$ is a function

$\mathcal{H} : \left(\bigcup_{i \in GEN-\{0\}} \mathcal{V}_i\right) \longrightarrow \left(\bigcup_{i \in GEN-\{0\}} A_i\right)$ such that $\mathcal{H}(\mathcal{V}_i) \subseteq A_i$ for all $i \in GEN$.

The *assignments* of variables are heterogeneous, i.e., they assign an element of the universe of genus i to a variable of genus i, respectively.

An *interpretation over a structure* $\mathcal{A}$ is a pair $\mathcal{I} = \langle \mathcal{A}, \mathcal{H} \rangle$ where $\mathcal{H}$ is a PHLM assignment of variables and $\mathcal{A}$ is a PHLM structure.

Notation. $\mathcal{I}^{a_i}_{x_i} = \langle \mathcal{A}, \mathcal{H}^{a_i}_{x_i} \rangle$ and $\mathcal{H}^{a_i}_{x_i} = (\mathcal{H} - \langle x_i, \mathcal{H}(x_i) \rangle) \cup \{\langle x_i, a_i \rangle\}$.
Given a structure $\mathcal{A}$ and a language $\mathcal{L}$ with signature Σ for PHLM. Given an interpretation $\mathcal{I}$ over $\mathcal{A}$, we define recursively *the denotation by $\mathcal{I}$ of an expression* ε of the language $\mathcal{L}$, noted $\mathcal{I}(\varepsilon)$.

1. $\mathcal{I}(x_i) = \mathcal{H}(x_i)$
2. $\mathcal{I}(f\varepsilon_1...\varepsilon_m) = f^{\mathcal{A}}(\mathcal{I}(\varepsilon_1), ..., \mathcal{I}(\varepsilon_m))$
3. $\mathcal{I}(r\varepsilon_1...\varepsilon_n) = r^{\mathcal{A}}(\mathcal{I}(\varepsilon_1), ..., \mathcal{I}(\varepsilon_m))$
4. $\mathcal{I}(\neg\varphi) = \neg^{\mathcal{A}}(\mathcal{I}(\varphi))$
5. $\mathcal{I}(\varphi \vee \psi) = \vee^{\mathcal{A}}(\mathcal{I}(\varphi), \mathcal{I}(\psi))$
6. $\mathcal{I}(V\varphi) = V^{\mathcal{A}}(\mathcal{I}(\varphi))$
7. $\mathcal{I}(\xi\varphi) = \xi^{\mathcal{A}}(\mathcal{I}(\varphi))$
8. $\mathcal{I}(\exists x_i \varphi) = \begin{cases} T \;\Leftrightarrow\; \{a \in A_i \mathbin{/} \mathcal{I}^a_x(\varphi) = T\} \neq \emptyset \\ F \;\Leftrightarrow\; \{a \in A_i \mathbin{/} \mathcal{I}^a_x(\varphi) = F\} = A_i \\ N \;\Leftrightarrow\; \begin{cases} \{a \in A_i \mathbin{/} \mathcal{I}^a_x(\varphi) = N\} \neq \emptyset \text{ and} \\ \{a \in A_i \mathbin{/} \mathcal{I}^a_x(\varphi) = T\} = \emptyset \end{cases} \end{cases}$
9. $\mathcal{I}(\tau_1 = \tau_2) = \begin{cases} T \;\Leftrightarrow\; \mathcal{I}(\tau_1) = \mathcal{I}(\tau_2) \\ F \;\Leftrightarrow\; \mathcal{I}(\tau_1) \neq \mathcal{I}(\tau_2) \end{cases}$
10. $\mathcal{I}(R\varepsilon_1...\varepsilon_n) = R^{\mathcal{A}}(\mathcal{I}(\varepsilon_1), ..., \mathcal{I}(\varepsilon_m))$

Now, it is easy to check that the following lemmas hold.

Lemma 1 ***(of Coincidence)***
Given a structure $\mathcal{A}$, and two assignments over $\mathcal{A}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, such that $\mathcal{H}_1(FREE(\varepsilon)) = \mathcal{H}_2(FREE(\varepsilon))$ for an expression ε, then $\langle \mathcal{A}, \mathcal{H}_1 \rangle(\varepsilon) = \langle \mathcal{A}, \mathcal{H}_2 \rangle(\varepsilon)$.

Lemma 2 ***(of Substitution)***
Given an interpretation $\mathcal{I} = \langle \mathcal{A}, \mathcal{H} \rangle$, for all $\varepsilon \in EXPR$ and all $\tau \in TERM$ holds $\mathcal{I}(S^{\tau}_{x_i}\varepsilon) = \mathcal{I}^{\mathcal{I}(\tau)}_{x_i}(\varepsilon)$.

If $\mathcal{I}$ is an interpretation for PHLM we have these two definitions corresponding to the concept of a model.

- $\mathcal{I}$ is a *strong model (s-model)* of a formula φ iff $\mathcal{I}(\varphi) = T$ ($\mathcal{I} \models_s \varphi$).
- $\mathcal{I}$ is a *weak model (w-model)* of a formula φ iff $\mathcal{I}(\varphi) \neq F$ ($\mathcal{I} \models_w \varphi$).

We also say that φ is *s-satisfiable* $(x \in \{s, w\})$ iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I}$ is an *x-model* of φ ($\mathcal{I} \models_x \varphi$).

A formula φ is *xy-logical consequence* $(x, y \in \{s, w\})$ of a set of formulas Γ (we note $\Gamma \models_{xy} \varphi$) iff for every interpretation $\mathcal{I}$, if $\mathcal{I}$ is a x-model of Γ then $\mathcal{I}$ is a y-model of φ.

To choose among the four possible consequences (ss, sw, ws, ww) which is the "best one" we compare them with the following theorem proved in [6].

Theorem 3 *If* $\Gamma \cup \{\varphi\} \subseteq FORM$ *then the following hold:*
$\Gamma \models_{sw} \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is NOT s-satisfiable*
$\Gamma \models_{ss} \varphi$ *iff* $\Gamma \cup \{\xi(\neg\varphi)\}$ *is NOT s-satisfiable*
$\Gamma \models_{ws} \varphi$ *iff* $\xi\Gamma \cup \{\xi(\neg\varphi)\}$ *is NOT s-satisfiable* ($\xi\Gamma = \{\xi\gamma \ / \ \gamma \in \Gamma\}$)
$\Gamma \models_{ws} \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is NOT w-satisfiable*
$\Gamma \models_{ww} \varphi$ *iff* $\Gamma \cup \{V(\neg\varphi)\}$ *is NOT w-satisfiable*
$\Gamma \models_{sw} \varphi$ *iff* $V\Gamma \cup \{V(\neg\varphi)\}$ *is NOT w-satisfiable* ($V\Gamma = \{V\gamma \ / \ \gamma \in \Gamma\}$

Theorem 4 *If* $\Gamma \cup \{\varphi\} \subseteq FORM$ *then the following hold:*

$$\Gamma \models_{ws} \varphi \Longrightarrow \left\{ \begin{array}{l} \Gamma \models_{ss} \varphi \\ \Gamma \models_{ww} \varphi \end{array} \right\} \Longrightarrow \Gamma \models_{sw} \varphi$$

Theorem 5 *If* $\Gamma \cup \{\varphi\} \subseteq FORM$ *then the following hold:*

$$\begin{array}{ccccc} & & \Gamma \models_{ss} \varphi & & \\ & & \Downarrow & & \\ \Gamma \models_{ws} \varphi \Longrightarrow V\Gamma \models_{ws} \xi\varphi & \Longleftrightarrow & \left\{ \begin{array}{l} \Gamma \models_{ss} \xi\varphi \\ V\Gamma \models_{ww} \varphi \end{array} \right\} & \Longleftrightarrow & \Gamma \models_{sw} \varphi \\ & & \Uparrow & & \\ & & \Gamma \models_{ww} \varphi & & \end{array}$$

The model-theoretic logical consequence for PHLM will be the one giving us the equivalence with the syntactical deduction for PHLM. To clarify this point we will define the properties of a logical consequence that are equivalent to the corresponding deductive calculus to possess some well known syntactical rules of inference. These important rules are Modus Ponens (MP), Generalization Rule (GEN) , Deduction Rule (DED) and Cut Rule (CUT).

Given a logical consequence relation $\models$, if $\Gamma \cup \Omega \cup \{\varphi, \psi\} \subseteq FORM$ then:
$\models$ verifies MP iff $\{\varphi,\ \varphi \rightarrow \psi\} \models \psi$
$\models$ verifies GEN iff $\varphi \models \forall x_i \varphi$ for $x_i \notin FREE(\varphi)$
$\models$ verifies DED iff $\Gamma \cup \{\varphi\} \models \psi$ then $\Gamma \models \varphi \rightarrow \psi$
$\models$ verifies CUT iff $\Gamma \models \varphi$ and $\Omega \cup \{\varphi\} \models \psi$ then $\Gamma \cup \Omega \models \psi$

Theorem 6 $\models_{ss}$ *verifies MP , CUT and GEN.*
$\models_{sw}$ *verifies MP , DED and GEN.*
$\models_{ww}$ *verifies DED, CUT and GEN.*
$\models_{ws}$ *verifies CUT.*

Thus $\models_{sw}$ is the only one having a good condition of satisfiability and verifying Modus Ponens, GEN and the deduction rule, it is also the weakest one.
For these reasons our choice for the PHLM semantics is the strong-weak logical consequence ($\models_{sw}$) and the strong satisfiability (s-satisfiability). Note that from now on we will write $\models$ instead of $\models_{sw}$, and $\models$ instead of $\models_s$.

3 PHLM Deductive Calculus

In the previous section the language and semantics for PHLM have been defined. We have chosen the strong-weak logical consequence ($\models_{sw}$), which according to the truth of premises guarantees the non-falsehood of conclusions. This consequence relation has the best properties with the only exception of not verifying the classical cut rule. And thus we find a limitation when we try to define a sound and complete deductive calculus for this consequence relation: if a logical consequence does not verify cut rule then it is not possible to find a sound and complete *Hilbert-style*[1] *calculus*[2]. The alternative is to look for a *Gentzen-style* calculus (a sequent calculus)[3], mainly formed by rules of inference. In [2], the first complete *Gentzen-type* for classical heterogeneous logic (there called manysorted logic) was presented.

Thus we look for a sequent calculus for the PHLM semantics defined before. What we want is to have the well proof theoretical properties of soundness and completeness. We are going to present one solution but other possibilities could be found.

3.1 Sequent calculus for PHLM

In what follows we will define a sequent calculus for PHLM. And we will prove that it is sound and complete with respect to the partial heterogeneous semantics using $\models_{sw}$.

Notation. In the following we will note $\models$ instead of $\models_{sw}$.

A *sequent* is a pair (Γ, Ω) of finite, possible empty, sets of formulas of the language. Γ is the *antecedent* and Ω is the *consequent* of the sequent. We use $\Gamma \vdash \Omega$ to denote the sequent (Γ, Ω).

[1] In [1], pp.34-37 Hilbert-type calculus is defined. A calculus with stress on the logical axioms and which uses the least rules of inference possible.
[2] In [6] this fact is proved.
[3] In [1], pp.37-40, Gentzen-type calculus is presented.

We also note $\Gamma_1, \Gamma_2 \vdash \Omega_1, \Omega_2$ and $\Gamma, \varphi \vdash \Omega, \psi$ in order to represent the sequents $\Gamma_1 \cup \Gamma_2 \vdash \Omega_1 \cup \Omega_2$ and $\Gamma \cup \{\varphi\} \vdash \Omega \cup \{\psi\}$ respectively, where $\Gamma_1, \Gamma_2, \Omega_1, \Omega_2, \Gamma$ and Ω are finite sets of formulas, and φ, ψ are formulas.

To define axioms and sequent rules (also known as deduction rules) of the sequent calculus for PHLM we consider the functionally complete set of connectives $\{\neg, V, \xi, \vee\}$, and the quantifier $\exists$.

Definition 7 *Axioms and sequents rules*

- **Axioms**

Identity	*ID:* $\varphi \vdash \varphi$
Introduction of V in the consequent	*IVC*: $\varphi \vdash V\varphi$
Introduction of ξ in the consequent	*IξC:* $\emptyset \vdash \xi\varphi$
Double negation-1	*DN*$_1$: $\varphi \vdash V\neg\neg\varphi$
Double negation-2	*DN*$_2$: $\neg V\neg\varphi \vdash \varphi$
Reflexivity (equality axiom)	*RE:* $\emptyset \vdash V(\tau = \tau)$

- **Structural rules**

Introd of hypothesis in the antecedent	*(IHA)* $\dfrac{\Gamma \vdash \Omega}{\Gamma, \varphi \vdash \Omega}$
Introd. of hypothesis in the consequent	*(IHC)* $\dfrac{\Gamma \vdash \Omega}{\Gamma \vdash \Omega, \varphi}$

- **Logical rules**

Introd. negation in the antecedent (1)	*(I¬A)* $\dfrac{\Gamma \vdash \Omega, \varphi}{\Gamma, \neg\varphi \vdash \Omega}$
Introd. negation in the consequent	*(I¬C)* $\dfrac{\Gamma, \varphi \vdash \Omega}{\Gamma \vdash \Omega, \neg\varphi}$
Introd. verification of negation in cons.	*(IV¬C)* $\dfrac{\Gamma, \varphi \vdash \Omega}{\Gamma \vdash \Omega, V\neg V\varphi}$

Introd. verification in the antecedent	*(IVA)* $\dfrac{\Gamma, \varphi \vdash \Omega}{\Gamma, V\varphi \vdash \Omega}$
Introd. verif. of verification in cons.	*(IVVC)* $\dfrac{\Gamma \vdash \Omega, V\varphi}{\Gamma \vdash \Omega, VV\varphi}$

Introd. difalsification in the antecedent	*(IξA)* $\dfrac{\Gamma \vdash \Omega, V\neg\varphi}{\Gamma,\ \xi\varphi\ \vdash\ \Omega}$
Introd. verif. of difalsification in cons.	*(IVξC)* $\dfrac{\Gamma, \neg\varphi \vdash \Omega}{\Gamma \vdash \Omega, V\xi\varphi}$

Introd. disj. in the ant.	(I∨A) $\dfrac{\Gamma_1, \varphi \vdash \Omega_1 \quad \Gamma_2, \psi \vdash \Omega_2}{\Gamma_1, \Gamma_2, \varphi \vee \psi \vdash \Omega_1, \Omega_2}$
Introd. disj. in cons.-1	(I∨C_1) $\dfrac{\Gamma \vdash \Omega, \varphi}{\Gamma \vdash \Omega, \varphi \vee \psi}$
Introd. disj. in cons.-2	(I∨C_2) $\dfrac{\Gamma \vdash \Omega, \psi}{\Gamma \vdash \Omega, \varphi \vee \psi}$
Introd. verif. of disj.-1	(IV∨C_1) $\dfrac{\Gamma_1 \vdash \Omega_1, V\varphi \quad \Gamma_2 \vdash \Omega_2, V\psi}{\Gamma_1, \Gamma_2 \vdash \Omega_1, \Omega_2, V(\varphi \vee \psi)}$
Introd. verif. of disj.-2	(IV∨C_2) $\dfrac{\Gamma_1 \vdash \Omega_1, V\varphi \quad \Gamma_2 \vdash \Omega_2, V\neg\psi}{\Gamma_1, \Gamma_2 \vdash \Omega_1, \Omega_2, V(\varphi \vee \psi)}$
Introd. verif. of disj.-3	(IV∨C_3) $\dfrac{\Gamma_1 \vdash \Omega_1, V\neg\varphi \quad \Gamma_2 \vdash \Omega_2, V\psi}{\Gamma_1, \Gamma_2 \vdash \Omega_1, \Omega_2, V(\varphi \vee \psi)}$
I. verif. neg. of disj. in c.	(IV¬∨C) $\dfrac{\Gamma_1 \vdash \Omega_1, V\neg\varphi \quad \Gamma_2 \vdash \Omega_2, V\neg\psi}{\Gamma_1, \Gamma_2 \vdash \Omega_1, \Omega_2, V\neg(\varphi \vee \psi)}$

- **Quantifier rules**

Intr. ∃ in ant.	(I∃A) $\dfrac{\Gamma, S_{x_i}^{y_i}\varphi \vdash \Omega}{\Gamma, \exists x_i\varphi \vdash \Omega}$, $y_i \notin FREE(\Gamma \cup \Omega \cup \{\exists x_i\varphi\})$
Intr. ∃ in cons.	(I∃C) $\dfrac{\Gamma \vdash \Omega, S_{x_i}^{\tau}\varphi}{\Gamma \vdash \Omega, \exists x_i\varphi}$, $\tau \in TERM_i$
Intr. verif. of ∃	(IV∃C) $\dfrac{\Gamma \vdash \Omega, VS_{x_i}^{\tau}\varphi}{\Gamma \vdash \Omega, V\exists x_i\varphi}$, $\tau \in TERM_i$

- **Cut rule.**[4]

$$(CUT)\ \frac{\Gamma_1 \vdash \Omega_1, V\varphi \quad \Gamma_2, \varphi \vdash \Omega_2}{\Gamma_1, \Gamma_2 \vdash \Omega_1, \Omega_2}$$

- **Equality rule**

Equals substitution	(SUB) $\dfrac{\Gamma \vdash \Omega, S_{x_i}^{\tau}\varphi}{\Gamma, \tau = \rho \vdash \Omega, S_{x_i}^{\rho}\varphi}$, *τ has the same genus as x_i iff ρ has the same genus as x_i*

[4] Observe that it is not the classical Cut rule.

3.2 Deductions in the calculus

A *derivation* (or *proof*) is a finite non-empty sequence of sequents

$$\Gamma_1 \vdash \Omega_1, ..., \ \Gamma_n \vdash \Omega_n$$

such that every $\Gamma_k \vdash \Omega_k$ is an axiom or it is obtained from one (or some) previous sequents in the sequence by applying one of the sequent (deduction) rules. In this case $\Gamma_n \vdash \Omega_n$ is called a *derivable sequent* of the calculus. We simply note $\Gamma_n \vdash \Omega_n$ in PHLM to indicate that there exist such a derivation.

A set of formulas Ω is *derivable (provable) from a set of formulas* Θ if there is a finite set $\Gamma \subseteq \Theta$, such that $\Gamma \vdash \Omega$ is derivable in the calculus. We will note it $\Theta \vdash \Omega$.

If $\emptyset \vdash \varphi$ we will say that φ is a *derivation* of the calculus. We will note $\vdash \varphi$. If $\emptyset \vdash V\varphi$ we will say that φ is a *theorem* of the calculus. We will note $\vdash V\varphi$.

Remark 1 . *In PHLM we have two different concepts, theorem and derivation, both corresponding to the classical concept of theorem.*

Now we will prove three derived rules of the calculus, which will go on to be very useful.

Theorem 8 *The following rules are derived rules of the calculus of PHLM.*

Elimination of $\neg$ *in the antecedent*	*(E¬A)*	$\dfrac{\Gamma, \neg\varphi \vdash \Omega}{\Gamma \vdash \Omega, \varphi}$
Elimination of $\neg$ *in the consequent*	*(E¬C)*	$\dfrac{\Gamma \vdash \Omega, \neg\varphi}{\Gamma, \varphi \vdash \Omega}$
Elimination of V *in the consequent*	*(EVC)*	$\dfrac{\Gamma \vdash \Omega, V\varphi}{\Gamma, \varphi \vdash \Omega}$

Proof.

(E¬A):

1. $\Gamma, \neg\varphi \vdash \Omega$	$[Premise]$
2. $\Gamma \vdash \Omega, V\neg V\neg\varphi$	$[IV\neg C, 1]$
3. $\neg V\neg\varphi \vdash \varphi$	$[Axiom\ DN_2]$
4. $\Gamma, \neg V\neg\varphi \vdash \varphi$	$[IHA, 3]$ -in fact successive applications of that rule
5. $\Gamma \vdash \Omega, \varphi$	$[CUT, 2, 4]$

(E¬C):

1. $\Gamma \vdash \Omega, \neg\varphi$	$[Premise]$
2. $\Gamma, \neg\neg\varphi \vdash \Omega$	$[I\neg A, 1]$
3. $\Gamma \vdash \Omega, V\neg V\neg\neg\varphi$	$[IV\neg C, 2]$
4. $\varphi \vdash V\neg\neg\varphi$	$[AxiomDN_1]$
5. $\varphi \vdash \Omega, V\neg\neg\varphi$	$[IHC, 4]$ -in fact successive applications of that rule
6. $\varphi, \neg V\neg\neg\varphi \vdash \Omega$	$[I\neg A, 5]$
7. $\Gamma, \varphi \vdash \Omega$	$[CUT, 3, 6]$

(EV C):

1. $\Gamma \vdash \Omega, V\varphi$ [*Premise*]
2. $\varphi \vdash \varphi$ [*Axiom ID*]
3. $\Gamma \vdash \Omega, \varphi$ [$CUT, 1, 2$]

■

4 PHLM Soundness

Next we will prove that the sequent calculus given in definition 7 is sound with respect to the PHLM semantics based on the strong-weak relation of logical consequence.

Definition 9 .

A sequent $\Gamma \vdash \Omega$ is *sound (with respect to the logical consequence relation* $\vDash$) iff $\Gamma \vDash \Omega$ (for all interpretation $\mathcal{I}$, if $\mathcal{I}(\varphi) = T$ for all $\varphi \in \Gamma$ then there is some $\psi \in \Omega$ such that $\mathcal{I}(\psi) \neq F$).

A sequent rule is *sound (with respect to the logical consequence relation* $\vDash$) iff it derives sound sequents from sound sequents.

4.1 Soundness of the calculus

Theorem 10 *Soundness of PHLM.*
For any $\Gamma \cup \Omega \subseteq FORM$, *if* $\Gamma \vdash \Omega$ *then* $\Gamma \vDash \Omega$.

Proof.
If $\Gamma \vdash \Omega$ then there is a finite set $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 \vdash \Omega$. It is enough to prove that $\Gamma_1 \vDash \Omega$.
We first prove the following lemma.

Lemma 11 *.The axioms and sequent rules of the deductive calculus for PHLM are sound (with respect to the consequence relation* $\vDash$*).*

Proof.
We will show that all the axioms are sound (that is, they are sound sequents) and that sequent rules are sound rules. We only present the proof for some rules, the remaining ones is an easy exercise.

Let $\mathcal{I}$ be any PHLM interpretation.

- (DN_1) The axiom $\varphi \vdash V\neg\neg\varphi$ is sound. $\mathcal{I}(\varphi) = T \Rightarrow \mathcal{I}(V\neg\neg\varphi) \neq F$ and then $\varphi \vDash V\neg\neg\varphi$.
- *(IHA)* The rule $\dfrac{\Gamma \vdash \Omega}{\Gamma, \varphi \vdash \Omega}$ is sound: suppose that $\mathcal{I}(\Gamma, \varphi) = T \Rightarrow \mathcal{I}(\Gamma) = T \Rightarrow \mathcal{I}(\Omega) \neq F$.

- The proof for the logical rules is a direct consequence from the semantical tables of the connectives.

- *(I∃A)* is sound. Suppose $\Gamma, S^{y_i}_{x_i}\varphi \vDash \Omega$ and that $\mathcal{I}(\Gamma, \exists x_i\varphi) = T$. So there is $a_i \in A_i$ *(A_i is the individual universe of genus i of $\mathcal{I}$)*, such that $\mathcal{I}^{a_i}_{x_i}\varphi = T$. Let $y_i \notin FREE(\Gamma \cup \Omega \cup \{\exists x_i\varphi\})$, we note $\mathcal{J} = \mathcal{I}^{a_i}_{y_i}$, observe that $\mathcal{J}$ agrees with $\mathcal{I}$ for all the variables except perhaps for y_i, and thus $\mathcal{J}$ agrees with $\mathcal{I}$ for the formulas non containing y_i as a free variable, then $\mathcal{J}(\Gamma) = \mathcal{I}(\Gamma) = T$ and $\mathcal{J}(S^{y_i}_{x_i}\varphi) = \mathcal{J}^{\mathcal{J}(y_i)}_{x_i}\varphi = \mathcal{J}^{a_i}_{x_i}\varphi = \mathcal{I}^{a_i}_{x_i}\varphi = T$. Thus $\mathcal{J}(\Gamma, S^{y_i}_{x_i}\varphi) = T$ and we obtain that there is a $\psi \in \Omega$ such that $\mathcal{I}(\psi) = \mathcal{J}(\psi) \neq F$. Thus the sequent $\Gamma, \exists x_i\varphi \vdash \Omega$ is sound.

- The CUT rule is sound. Suppose that $\Gamma_1 \vDash \Omega_1, V\varphi$ and $\Gamma_2, \varphi \vDash \Omega_2$, and suppose that $\mathcal{I}(\Gamma_1, \Gamma_2) = T$. From the hypothesis there is $\psi \in \Omega_1$ such that $\mathcal{I}(\psi) \neq F$ or $\mathcal{I}(V\varphi) \neq F$. But in the latter case $\mathcal{I}(\varphi) = T$. From $\Gamma_2, \varphi \vDash \Omega_2$ we obtain $\phi \in \Omega_2$ such that $\mathcal{I}(\phi) \neq F$. Thus $\Gamma_1, \Gamma_2 \vDash \Omega_1, \Omega_2$.

- The equality axiom and rule are consequence of the interpretation of equality being the identity in the domain of genus i.■

Now we can finish the proof of the soundness theorem:
Suppose that there is a proof $\Theta_1 \vdash \Psi_1, \Theta_2 \vdash \Psi_2$, ... ,$\Gamma_1 \vdash \Omega$. We conclude that $\Gamma_1 \vDash \Omega$ because all the axioms and rules are sound.
■

5 Consistency

In this section we will define the syntactical concept of consistency of a set of formulas corresponding to the semantical concept of satisfiability.

$\Gamma \subseteq FORM$ is **contradictory** iff for all $\varphi \in FORM$, $\Gamma \vdash \varphi$. We note it $\Gamma \vdash \perp$

$\Gamma \subseteq FORM$ is **consistent** iff Γ is not contradictory.

$\Gamma \subseteq FORM$ is **maximally consistent** iff Γ is consistent and for all $\varphi \in FORM$, if $\varphi \notin \Gamma$ then $\Gamma \cup \{\varphi\}$ is contradictory.

$\Gamma \subseteq FORM$ **contains witnesses** iff for all existential formula:

1. if $\exists x_i\varphi \in \Gamma$ then $S^{\tau_i}_{x_i}\varphi \in \Gamma$ for a term τ_i with the same genus as x_i , and
2. if $\neg\exists x_i\varphi \notin \Gamma$ then $S^{\tau_i}_{x_i}\varphi \in \Gamma$ or $\neg S^{\tau_i}_{x_i}\varphi \notin \Gamma$ for a term τ_i with the same genus as x_i

5.1 Theorems of consistency

Theorem 12 . *If $\Gamma \cup \{\varphi\} \subseteq FORM$, the following hold:*
1. If Γ is consistent and $\Gamma` \subseteq \Gamma$, then $\Gamma`$ is also consistent.
2. If Γ is contradictory and $\Gamma \subseteq \Gamma`$, then $\Gamma`$ is also contradictory.
3. If Γ is consistent, $\exists x_i\varphi \in \Gamma$ and $z_i \notin FREE(\Gamma)$, then $\Gamma \cup \{S^{z_i}_{x_i}\varphi\}$ is consistent.
4. If Γ has a (strong) model then Γ is consistent.

Proof.
1. and 2. are direct consequence of the definitions.
3. If $\Gamma \cup \{S^{z_i}_{x_i}\varphi\} \vdash \psi$ for all $\psi \in FORM$, then we obtain $\Gamma = \Gamma \cup \{\exists x_i\varphi\} \vdash \psi$ (by $I\exists A$) for all $\psi \in FORM$, in contradiction with the consistency of Γ.
4. If Γ is contradictory then $\Gamma \vdash \psi$ for all $\psi \in FORM$ and by the soundness theorem, $\Gamma \vDash \psi$ for all $\psi \in FORM$. In particular $\Gamma \vDash V\varphi$ and $\Gamma \vDash V\neg\varphi$ for some $\varphi \in FORM$. Now consider $\mathcal{I}$ is a strong model of Γ, then $\mathcal{I}(\Gamma) = T$, thus $\mathcal{I}(V\varphi) \neq F$ and $\mathcal{I}(V\neg\varphi) \neq F$ which is impossible.
■

5.2 Theorems of Maximal Consistency

Theorem 13 . *If $\Gamma \subseteq FORM$ is a maximally consistent set, and $\varphi, \psi \in FORM$, the following hold:*

1. *If* $\Gamma \vdash V\varphi \Rightarrow \varphi \in \Gamma$
2. $V\varphi \in \Gamma$ *iff* $\varphi \in \Gamma$
3. *If* $\varphi \in \Gamma$ *then* $\neg\varphi \notin \Gamma$
4. $\varphi \in \Gamma$ *iff* $\neg\neg\varphi \in \Gamma$
5. $\varphi \notin \Gamma$ *iff* $\neg V\varphi \in \Gamma$
6. $\xi\varphi \in \Gamma$ *iff* $\varphi \in \Gamma$ *or* ($V\varphi \notin \Gamma$ *and* $V\neg\varphi \notin \Gamma$)
7. $\neg\xi\varphi \notin \Gamma$
8. $\varphi \vee \psi \in \Gamma$ *iff* ($\varphi \in \Gamma$ *and* ($V\psi \in \Gamma$ *or* $V\neg\psi \in \Gamma$)) *or* ($\psi \in \Gamma$ *and* ($V\varphi \in \Gamma$ *or* $V\neg\varphi \in \Gamma$))[5]
9. $\neg(\varphi \vee \psi) \in \Gamma$ *iff* $\neg\varphi \in \Gamma$ *and* $\neg\psi \in \Gamma$
10. *If Γ contains witnesses, then $\exists x_i\varphi \in \Gamma$ iff there is $\sigma_i \in TERM_i$ with the same genus as x_i such that $S^{\sigma_i}_{x_i}\varphi \in \Gamma$*
11. *If Γ contains witnesses, then $\neg\exists x_i\varphi \in \Gamma$ iff for all $\tau_i \in TERM_i$ with the same genus as x_i, $S^{\tau_i}_{x_i}\varphi \notin \Gamma$ and $\neg S^{\tau_i}_{x_i}\varphi \in \Gamma$*

[5] As a consequence if $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Proof.
1. Suppose that $\Gamma \vdash V\varphi$, and that $\varphi \notin \Gamma$. $\Gamma, \varphi \vdash \bot$ because of maximal consistency, and then $\Gamma \vdash \bot$ by *CUT* rule, which is a contradiction.

2. One direction is trivial from 1. For the other direction suppose that $\varphi \in \Gamma$ and $V\varphi \notin \Gamma$, then $\Gamma \vdash VV\varphi$ (by axiom *IVC* and rule *IVVC)* . $\Gamma, V\varphi \vdash \bot$, so we obtain a contradiction by using *CUT*.

3. Suppose that $\varphi \in \Gamma$ then $\Gamma \vdash V\varphi$. Suppose that $\neg\varphi \in \Gamma$, then $\Gamma \vdash \neg\varphi, \psi$ for all formula ψ *(IHC)*. We obtain $\Gamma, \varphi \vdash \psi$ for all formula ψ ($E\neg C$). Finally $\Gamma \vdash \bot$ (by *CUT* rule).

4. ($\Rightarrow$) Suppose that $\varphi \in \Gamma$. If $\neg\neg\varphi \notin \Gamma$ then $\Gamma, \neg\neg\varphi \vdash \bot$ and $\Gamma, \varphi \vdash \bot$ *($E\neg A$* and *$E\neg C$)*, in contradiction with the consistency of Γ.
($\Leftarrow$) Suppose that $\neg\neg\varphi \in \Gamma$. If $\varphi \notin \Gamma$ then $\Gamma, \varphi \vdash \bot$, and $\Gamma, \neg\neg\varphi \vdash \bot$ *($I\neg C$*, and *$I\neg A$)*. So we get a contradiction.

5. ($\Rightarrow$) If $\varphi \notin \Gamma$ then $\Gamma, \varphi \vdash \bot$. Suppose that $\neg V\varphi \notin \Gamma$, then $\Gamma, \neg V\varphi \vdash \bot$ and $\Gamma \vdash V\varphi, \bot$ by *($E\neg A$)*. $\Gamma \vdash \bot$ by *CUT.*
($\Leftarrow$) If $\neg V\varphi \in \Gamma$ then $\neg\neg V\varphi \notin \Gamma$ by 3, and $V\varphi \notin \Gamma$ by 4. So $\varphi \notin \Gamma$ by 2.

6.($\Rightarrow$) Suppose that $\xi\varphi \in \Gamma$. Suppose $\varphi \notin \Gamma$, so $\Gamma, \varphi \vdash \bot$. Suppose that $V\varphi \in \Gamma$ (it is impossible) or $V\neg\varphi \in \Gamma$. $\Gamma \vdash V\neg\varphi$ and $\Gamma \vdash V\neg\varphi, \bot$ *(IHC)* thus $\Gamma, \xi\varphi \vdash \bot$ by $I\xi A$. Contradiction.
($\Leftarrow$) Suppose that $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$ and $\Gamma, \neg\varphi \vdash \bot$ *($I\neg A$* and *IHC).* Thus $\Gamma \vdash V\xi\varphi, \bot$ *($IV\xi C$)* If $\xi\varphi \notin \Gamma$ we get a contradiction. Now suppose that $V\varphi \notin \Gamma$ and $V\neg\varphi \notin \Gamma$, then $\neg\varphi \notin \Gamma$ and $\Gamma, \neg\varphi \vdash \bot$, so $\Gamma \vdash \bot, V\xi\varphi$ *($IV\xi C$)*. Finally, if $\xi\varphi \notin \Gamma$, then $\Gamma, \xi\varphi \vdash \bot$ and we get a contradiction by using *CUT.*

7. If $\neg\xi\varphi \in \Gamma$ then $\Gamma \vdash V\neg\xi\varphi$ by 2. Also $\Gamma \vdash \xi\varphi, \bot$ by ($I\xi C$) and *(IHC).* Thus $\Gamma, \neg\xi\varphi \vdash \bot$ by ($I\neg A$) and $\Gamma \vdash \bot$ *(CUT)* which is a contradiction.

8. ($\Rightarrow$) Suppose that $\varphi \vee \psi \in \Gamma$. We first prove that $\varphi \in \Gamma$ or $\psi \in \Gamma$. Suppose that $\varphi \notin \Gamma$ and $\psi \notin \Gamma$, then $\Gamma, \varphi \vdash \bot$ and $\Gamma, \psi \vdash \bot$, by *($I \vee A$)* we obtain $\Gamma, \varphi \vee \psi \vdash \bot$, then $\Gamma \vdash \bot$ in contradiction with Γ consistent.
Now, suppose that $\varphi \vee \psi \in \Gamma$ and $\varphi \in \Gamma$ and $V\psi \notin \Gamma$ and $V\neg\psi \notin \Gamma$, so $\psi \notin \Gamma$ and then $\Gamma, \psi \vdash \bot$ and thus $\Gamma, \varphi \vee \psi \vdash \bot$ *($I\vee A$)* again. It is a contradiction. The other case is similar.
($\Leftarrow$) Suppose that $\varphi \in \Gamma$ and ($V\psi \in \Gamma$ or $V\neg\psi \in \Gamma$). In the case $V\psi \in \Gamma$, $\Gamma \vdash V\psi$ and $\Gamma \vdash V\varphi$, so $\Gamma \vdash V(\varphi \vee \psi)$ *($IV\vee C_1$)*. The other case is similar by using $IV\vee C_2$.

9.($\Rightarrow$) Suppose that $\neg(\varphi \vee \psi) \in \Gamma$, then $\Gamma \vdash V\neg(\varphi \vee \psi)$. Suppose that $\neg\varphi \notin \Gamma$, then $\Gamma, \neg\varphi \vdash \bot$, and $\Gamma \vdash \bot, \varphi$ by ($E\neg A$). Thus $\Gamma \vdash \bot, \varphi \vee \psi$ by ($I\vee C_1$). So $\Gamma, \neg(\varphi \vee \psi) \vdash \bot$ by ($I\neg A$),we get a contradiction by *CUT*. The other case is similar with ($I\vee C_2$).
($\Leftarrow$) If $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$ then $\Gamma \vdash V\neg\varphi$ and $\Gamma \vdash V\neg\psi$. Thus $\Gamma \vdash V\neg(\varphi \vee \psi)$ by ($IV\neg \vee C$) and $\neg(\varphi \vee \psi) \in \Gamma$.

10. ($\Rightarrow$) Suppose that $\exists x_i\varphi \in \Gamma$. Then there is a term σ_i with the same genus as x_i such that $S^{\sigma_i}_{x_i}\varphi \in \Gamma$, by definition of containing witnesses.

($\Leftarrow$) If there is a term τ_i with the same genus as x_i such that $S^{\tau_i}_{x_i}\varphi \in \Gamma$ then $\Gamma \vdash$, $V\exists x_i\varphi$ by $(IV\exists C)$. Suppose that $\exists x_i\varphi \notin \Gamma$ then $\Gamma, \exists x_i\varphi \vdash \perp$. A contradiction by using the CUT.

11. ($\Rightarrow$) Suppose that $\neg\exists x_i\varphi \in \Gamma$, then $\Gamma \vdash V\neg\exists x_i\varphi$. If there is a term $\tau_i \in TERM_i$ such that $S^{\tau_i}_{x_i}\varphi \in \Gamma$ or $\neg S^{\tau_i}_{x_i}\varphi \notin \Gamma$, then $\Gamma \vdash S^{\tau_i}_{x_i}\varphi$ or $\Gamma, \neg S^{\tau_i}_{x_i}\varphi \vdash \perp$. In the second case $\Gamma, \neg S^{\tau_i}_{x_i}\varphi \vdash S^{\tau_i}_{x_i}\varphi$ in particular, and $\Gamma \vdash S^{\tau_i}_{x_i}\varphi$ by $(E\neg A)$. Then $\Gamma \vdash \exists x_i\varphi$ by $(I\exists C)$ and $\Gamma \vdash \exists x_i\varphi, \perp$ by *(IHC),* so $\Gamma, \neg\exists x_i\varphi \vdash \perp$. Finally $\Gamma \vdash \perp$ *(CUT)* in contradiction with the consistency of Γ.
($\Leftarrow$) If $\neg\exists x_i\varphi \notin \Gamma$ then there is $\sigma_i \in TERM_i$ such that $S^{\sigma_i}_{x_i}\varphi \in \Gamma$ or $\neg S^{\sigma_i}_{x_i}\varphi \notin \Gamma$, by the existence of witnesses.
■

5.3 Consistency and Finiteness

Lemma 14 ***(Finiteness of Derivability)***
If $\Gamma \subseteq FORM$, $\Gamma \vdash \Omega$ *iff there is a finite subset* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \vdash \Omega$.

Proof.
It is a consequence of the definition of a proof.
■

Lemma 15 ***(Finiteness of Consistency)***
If $\Gamma \subseteq FORM$, Γ *is consistent iff every finite subset of* Γ *is consistent.*

Proof.
($\Leftarrow$) Suppose that every finite subset of Γ is consistent. If Γ was contradictory then $\Gamma \vdash \gamma$ for all $\gamma \in FORM$, and by lemma 14 there is a contradictory finite subset of Γ, in contradiction with our hypothesis.
■

6 PHLM Completeness

Now we can prove that the sequent calculus based on the rules of 7 is strongly complete with respect to the strong-weak consequence and that it is weakly complete with respect to the strong validity.

We only consider the case of countable PHLM languages without equality (its set of formulas is countable, and there is not a symbol for equality). The proof for non-countable languages is similar although it needs the axiom of choice. The proof for languages with equality will be provided later on.

6.1 Strong completeness of the calculus

Theorem 16 *(Strong completeness for PHLM)*
For any $\Gamma \cup \Omega \subseteq FORM$, *if* $\Gamma \vDash \Omega$ *then* $\Gamma \vdash \Omega$.

Proof.
We will present a Henkin-style proof, so we first prove the following 5 propositions:

Lemma 17 *If* Γ *is consistent and FREE*(Γ) *is finite, then there is a maximally consistent set* Γ^* *with witnesses such that* $\Gamma \subseteq \Gamma^*$.

Lemma 18 ***(Henkin)****. If* Γ^* *is maximally consistent with witnesses, then* Γ^* *has a strong model (countable if the language is countable).*

Corollary 19 *. If* Γ *is consistent and FREE*(Γ) *is finite, then* Γ *has a strong model (countable if the language is countable).*

Lemma 20 *If* $\Gamma \subseteq FORM(\mathcal{L})$*, let* $\mathcal{L}_\Gamma^+$ *be the language obtained by adding to* $\mathcal{L}$ *a new constant for every free variable occurring in* Γ*. Let* Γ^+ *be the set of closed formulas of* $\mathcal{L}_\Gamma^+$ *obtained by systematically replacing every free variable with the corresponding new constant. Given* $\Gamma \subseteq FORM(\mathcal{L})$ *consistent, if* $\Gamma^+ \subseteq SENT(\mathcal{L}_\Gamma^+)$ *has a countable model, then* Γ *also has a countable model.*

Theorem 21 ***(Henkin).*** *If* Γ *is consistent then* Γ *has a strong model.*

The proofs of these 5 propositions will follow the path:
(1) Lemma 17,
(2) Lemma 18,
(3) Lemma 17 and lemma 18 $\Rightarrow$ corollary 19,
(4) Lemma 20,
(5) Corollary 19 and lemma 20 $\Rightarrow$ theorem 21.

Finally the *(strong) completeness theorem* is a consequence of the theorem 21 (Henkin):
Suppose that $\Gamma \models \Omega$, then $\Gamma \vDash \varphi$ for some $\varphi \in \Omega$. $\Gamma \cup \{\neg\varphi\}$ is s-unsatisfiable by the satisfiability lemma, that is, it has no strong model. By Henkin theorem we conclude that $\Gamma \cup \{\neg\varphi\}$ is contradictory, then $\Gamma, \neg\varphi \vdash \varphi$ in particular, and so $\Gamma \vdash \varphi$ $(E\neg A)$. Finally $\Gamma \vdash \Omega$ *(IHC)*.
■

Proof of Lemma 17 (Lindenbaum lemma).
Let φ_1, φ_2, φ_3, ... be a numeration[6] of the members of FORM. We also need a numeration of the variables of each genus: let x_{i1}, x_{i2}, x_{i3}, ... be a numeration of the variables of genus i. We can build a family of subsets of FORM $\{\Gamma_n\}_{n\in\omega}$ by induction:

- $\Gamma_0 = \Gamma$

[6] If the language was non-countable this condition would change and we should give a construction by using transfinite recursion.

- Assume that Γ_n has been built already,

$$\Gamma_{n+1} = \begin{cases} \Gamma_n, \text{ if } \Gamma_n \cup \{\varphi_n\} \text{ is contradictory} \\ \Gamma_n \cup \{\varphi_n\}, \text{ if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent and} \\ \qquad \varphi_n \text{ is not an existential formula} \\ \Gamma_n \cup \{\varphi_n, S_{x_i}^{y_{ip}}\psi\}, \text{ if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent, } \varphi_n \equiv \exists x_i \psi \\ \qquad \text{and } y_{ip} \notin FREE(\Gamma_n \cup \{\varphi_n\}), \\ \qquad \text{with } p \text{ the least possible} \end{cases}$$

Lemma.

1. For every $n \in \omega$, Γ_n is consistent.
2. $\Gamma^* = \bigcup_{n\in\omega} \Gamma_n$ is consistent.
3. $\Gamma \subseteq \Gamma^*$.
4. Γ^* is maximally consistent.
5. Γ^* contain witnesses.

Proof of lemma.

1. The proof is by induction.
   $\Gamma_0 = \Gamma$ is consistent by the hypothesis,
   Suppose that Γ_n is consistent. For Γ_{n+1} we have three possibilities:
   i) $\Gamma_{n+1} = \Gamma_n$ and it is consistent by the inductive hypothesis.
   ii) $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ and it is consistent by construction.
   iii) $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n, S_{x_i}^{y_{ip}}\psi\}$, where $\varphi_n \equiv \exists x_i\psi$ and $y_{ip} \notin FREE(\Gamma_n \cup \{\varphi_n\})$ and $\Gamma_n \cup \{\varphi_n\}$ is consistent. Then, by theorem 12.3, Γ_{n+1} is consistent. We required that in Γ only a finite number of variables appear, to ensure that we will always have new ones to use in the substitution.

2. Suppose that Γ^* is contradictory, by lemma on finiteness of consistency, there must be a finite subset of Γ^*, say Ψ, such that Ψ is contradictory. Let $C = \{n \in \omega \;/ \Gamma_n$ appears in the construction of Γ^*, and such that $\Psi \subseteq \Gamma_n\}$. Given that Ψ is finite, we know that $C \neq \emptyset$. Let p be the minimum of C, then $\Psi \subseteq \Gamma_p$ and by consistency theorem 12.2, Γ_p is also contradictory.

3. Obviously.

4. If $\varphi \in FORM - \Gamma^*$, we will see that $\Gamma^* \cup \{\varphi\}$ is contradictory. Recall that we have a numeration of FORM, and thus φ must have a place in this ordering, let us suppose that $\varphi = \varphi_m$. We know that $\varphi_m \notin \Gamma_{m+1}$, its natural place if $\varphi_m \in \Gamma^*$. The only possible reason is that $\Gamma_m \cup \{\varphi_m\}$ is contradictory. Therefore, $\Gamma^* \cup \{\varphi_m\}$ is contradictory as well, because it contains a contradictory subset.

5. Suppose that $\exists x_i\psi \in \Gamma^*$. As above, $\exists x_i\psi \equiv \varphi_n$ in the numeration of FORM, and $\varphi_n \in \Gamma_{n+1}$. Since Γ^* is consistent, $\Gamma_n \cup \{\varphi_n\}$ is consistent too, and according to our construction $\Gamma_{n+1} = \Gamma_n \cup \{\exists x_i\psi, S_{x_i}^{y_{ip}}\psi\}$.

■

Proof of Lemma 18 (Henkin lemma).

Suppose that Γ^* is maximally consistent and containing witnesses. We are going to construct a strong model of Γ^* . It will be the model described by the formulas in Γ^*.

Consider $\mathcal{B} = \langle \{B_i\}_{i \in GEN}\ , \{f^{\mathcal{B}}\}_{f \in OPER.SYM} \rangle$, where

1. $B_i = TERM_i$ is the set of terms of genus i , if $i \neq 0$, and $B_0 = \{T, F, N\}$

2. For each $f \in OPER.SYM$ with RANG$(f) = \langle i_0, i_1, ..., i_n \rangle$ with $i_0 \neq 0$, $f^{\mathcal{B}}(\tau_1, ..., \tau_n) = f\tau_1...\tau_n$, for all sequences of terms $\tau_1, ..., \tau_n$ with τ_j of genus i_j, for each $j \in \{1, ..., n\}$.

3. For each $r \in OPER.SYM$ with RANG$(r) = \langle 0, i_1, ..., i_n \rangle$ with $i_j \neq 0$, and for all sequences of terms $\tau_1, ..., \tau_n$ where for any $j \in \{1, ..., n\}$, τ_j has genus i_j:
$r^{\mathcal{B}}(\tau_1, ..., \tau_n) = T$ iff $r\tau_1...\tau_n \in \Gamma^*$
$r^{\mathcal{B}}(\tau_1, ..., \tau_n) = F$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \in \Gamma^*$
$r^{\mathcal{B}}(\tau_1, ..., \tau_n) = N$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \notin \Gamma^*$

4. For each $r \in OPER.SIM$ such that RANG$(r) = n$:
$r^{\mathcal{B}}(\tau_1, ..., \tau_n) = T$ iff $r\tau_1...\tau_n \in \Gamma^*$
$r^{\mathcal{B}}(\tau_1, ..., \tau_n) = F$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \in \Gamma^*$
$r^{\mathcal{B}}(\tau_1, ..., \tau_n) = N$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \notin \Gamma^*$

5. Besides that, we want the connectives being standard; that is, $\neg^{\mathcal{B}}$, $\vee^{\mathcal{B}}$, $V^{\mathcal{B}}$ and $\xi^{\mathcal{B}}$ being standard.

Now we define a variable assignment and an interpretation:
Let $\mathcal{I} = \langle \mathcal{B}, \mathcal{H} \rangle$ be an interpretation, with $\mathcal{H} : \bigcup_{i \in GEN - \{0\}} \mathcal{V}_i \longrightarrow \bigcup_{i \in GEN - \{0\}} B_i$

$$x_i \longrightarrow x_i$$

We finally prove that $\mathcal{I}$ is a strong model of Γ^* ($\mathcal{I}(\varphi) = T$ for all $\varphi \in \Gamma^*$). In fact we will prove even more:

Lemma.
(a) *For each* $\tau \in TERM$, $I(\tau) = \tau$.
(b) *For each* $\varphi \in FORM$:
$I(\varphi) = T$ *iff* $\varphi \in \Gamma^*$
$I(\varphi) = F$ *iff* $\varphi \notin \Gamma^*$ *and* $\neg\varphi \in \Gamma^*$
$I(\varphi) = N$ *iff* $\varphi \notin \Gamma^*$ *and* $\neg\varphi \notin \Gamma^*$

Proof of lemma.
(a) The proof for terms is by induction on the term length.
$\mathcal{I}(x_i) = \mathcal{H}(x_i) = x_i$
$\mathcal{I}(f\tau_1...\tau_n) = f^{\mathcal{B}}(\mathcal{I}(\tau_1), ..., \mathcal{I}(\tau_n)) = f^{\mathcal{B}}(\tau_1, ...\tau_n)$.

(Where f is a function symbol of type $\langle i_0, ..., i_n\rangle$ with $i_0 \neq 0$). We use the induction hypothesis on the terms $\tau_k (1 \leq k \leq n)$ of genus i_k.

(b) The proof for formulas is by induction on the number of occurrences of $\neg$, V, ξ , $\vee$ and $\exists$ in the construction of the formula, .

1. If $\varphi \equiv r\tau_1...\tau_n$ with r a relation symbol of type $\langle 0, i_1, ..., i_n\rangle$ with $i_1 \neq 0$ or r a relation symbol of ariety n, by using (3) and (4):

   $\mathcal{I}(\varphi) = T$ iff $\mathcal{I}(r\tau_1...\tau_n) = r^{\mathcal{B}}(\tau_1, ..., \tau_n) = T$ iff $r\tau_1...\tau_n \in \Gamma^*$.

   $\mathcal{I}(\varphi) = F$ iff $r^{\mathcal{B}}(\tau_1, ..., \tau_n) = F$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \in \Gamma^*$

   $\mathcal{I}(\varphi) = N$ iff $r^{\mathcal{B}}(\tau_1, ..., \tau_n) = N$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \notin \Gamma^*$.

2. If $\varphi \equiv \neg\psi$.

   $\mathcal{I}(\neg\psi) = T$ iff $\mathcal{I}(\psi) = F$ iff $\psi \notin \Gamma^*$ and $\neg\psi \in \Gamma^*$ (by induction hypothesis) iff $\neg\psi \in \Gamma^*$ (by 13.3 and 13.4).

   $\mathcal{I}(\neg\psi) = F$ iff $\mathcal{I}(\psi) = T$ iff $\psi \in \Gamma^*$ (ind. hyp.) iff $\neg\psi \notin \Gamma^*$ and $\neg\neg\psi \in \Gamma^*$ (by 13.3 and 13.4).

   $\mathcal{I}(\neg\psi) = N$ iff $\mathcal{I}(\psi) = N$ iff $\psi \notin \Gamma^*$ and $\neg\psi \notin \Gamma^*$ (ind. hyp.) iff $\neg\psi \notin \Gamma^*$ and $\neg\neg\psi \notin \Gamma^*$ (by 13.4).

3. If $\varphi \equiv V\psi$.

   $\mathcal{I}(V\psi) = T$ iff $\mathcal{I}(\psi) = T$ iff $\psi \in \Gamma^*$ (ind. hyp.) iff $V\psi \in \Gamma^*$ (by 13.2).

   $\mathcal{I}(V\psi) = F$ iff $\mathcal{I}(\psi) = F$ or $\mathcal{I}(\psi) = N$ iff ($\psi \notin \Gamma^*$ and $\neg\psi \in \Gamma^*$) or ($\psi \notin \Gamma^*$ and $\neg\psi \notin \Gamma^*$) (ind. hyp.) iff $\psi \notin \Gamma^*$ iff $V\psi \notin \Gamma^*$ and $\neg V\psi \in \Gamma^*$ (by 13.2 and 13.5).

   $\mathcal{I}(V\psi) = N$ is not possible.

4. If $\varphi \equiv \xi\psi$.

   $\mathcal{I}(\xi\psi) = T$ iff $\mathcal{I}(\psi) = T$ or $\mathcal{I}(\psi) = N$ iff $\psi \in \Gamma^*$ or ($\psi \notin \Gamma^*$ and $\neg\psi \notin \Gamma^*$) (ind. hyp.) iff $\psi \in \Gamma^*$ or ($V\psi \notin \Gamma^*$ and $V\neg\psi \notin \Gamma^*$) (by 13.2) iff $\xi\psi \in \Gamma^*$ (by 13.6).

   $\mathcal{I}(\xi\psi) = F$ is an impossible case.

   $\mathcal{I}(\xi\psi) = N$ iff $\mathcal{I}(\psi) = F$ iff $\psi \notin \Gamma^*$ and $\neg\psi \in \Gamma^*$ (ind. hyp.) iff $\psi \notin \Gamma^*$ and $V\neg\psi \in \Gamma^*$ (by 13.2) iff $\xi\psi \notin \Gamma^*$ (by 13.6) iff $\xi\psi \notin \Gamma^*$ and $\neg\xi\psi \notin \Gamma^*$ (by 13.7).

5. If $\varphi \equiv \alpha \vee \beta$.

   $\mathcal{I}(\alpha \vee \beta) = T$ iff ($\mathcal{I}(\alpha) = T$ and $\mathcal{I}(\beta) \neq N$) or ($\mathcal{I}(\alpha) \neq N$ and $\mathcal{I}(\beta) = T$) iff ($\alpha \in \Gamma^*$ and, $\beta \in \Gamma^*$ or $\neg\beta \in \Gamma^*$) or ($\alpha \in \Gamma^*$ or $\neg\alpha \in \Gamma^*$ and, $\beta \in \Gamma^*$) (ind. hyp.) iff ($\alpha \in \Gamma^*$ and, $V\beta \in \Gamma^*$ or $V\neg\beta \in \Gamma^*$) or ($V\alpha \in \Gamma^*$ or $V\neg\alpha \in \Gamma^*$ and, $\beta \in \Gamma^*$) (by 13.2) iff $\alpha \vee \beta \in \Gamma^*$ (by 13.8)

   $\mathcal{I}(\alpha \vee \beta) = F$ iff $\mathcal{I}(\alpha) = F$ and $\mathcal{I}(\beta) = F$ iff ($\alpha \notin \Gamma^*$ and $\neg\alpha \in \Gamma^*$) and ($\beta \notin \Gamma^*$ and $\neg\beta \in \Gamma^*$) (ind. hyp.) iff ($\alpha \notin \Gamma^*$ and $\beta \notin \Gamma^*$) and ($\neg\alpha \in \Gamma^*$ and $\neg\beta \in \Gamma^*$) iff $\alpha \vee \beta \notin \Gamma^*$ and $\neg(\alpha \vee \beta) \in \Gamma^*$ (by 13.8 and 13.9).

$\mathcal{I}(\alpha \vee \beta) = N$ iff $\mathcal{I}(\alpha) = N$ or $\mathcal{I}(\beta) = N$ iff ($\alpha \notin \Gamma^*$ and $\neg\alpha \notin \Gamma^*$) or ($\beta \notin \Gamma^*$ and $\neg\beta \notin \Gamma^*$) (ind. hyp.) iff (($\alpha \notin \Gamma^*$ or ($\beta \notin \Gamma^*$ and $\neg\beta \notin \Gamma^*$) and ($\beta \notin \Gamma^*$ or ($\alpha \notin \Gamma^*$ and $\neg\alpha \notin \Gamma^*$)) and ($\neg\alpha \notin \Gamma^*$ or $\neg\beta \notin \Gamma^*$) iff (($\alpha \notin \Gamma^*$ or ($V\beta \notin \Gamma^*$ and $V\neg\beta \notin \Gamma^*$)) and ($\beta \notin \Gamma^*$ or ($V\alpha \notin \Gamma^*$ and $V\neg\alpha \notin \Gamma^*$)) and ($\neg\alpha \notin \Gamma^*$ or $\neg\beta \notin \Gamma^*$) (by 13.2) iff $\alpha \vee \beta \notin \Gamma^*$ and $\neg(\alpha \vee \beta) \notin \Gamma^*$ (by 13.8 and 13.9).

6. If $\varphi \equiv \exists x_i \varphi$.

   $\mathcal{I}(\exists x_i \varphi) = T$ iff there is $\sigma_i \in B_i$ such that $\mathcal{I}(S^{\sigma_i}_{x_i}(\varphi)) = T$ (by substitution lemma) iff there is $\sigma_i \in B_i$ such that $S^{\sigma_i}_{x_i}(\varphi) \in \Gamma^*$ (ind. hyp.) iff $\exists x_i \varphi \in \Gamma^*$ (by 13.10).

   $\mathcal{I}(\exists x \varphi) = F$ iff for all $\tau_i \in B_i$, $\mathcal{I}(S^{\tau_i}_{x_i}(\varphi)) = F$ (by substitution lemma) iff for all $\tau_i \in B_i$, $S^{\tau_i}_{x_i}(\varphi) \notin \Gamma^*$ and $\neg S^{\tau_i}_{x_i}(\varphi) \in \Gamma^*$ (ind. hyp.) iff $\exists x_i \varphi \notin \Gamma^*$ and $\neg\exists x_i \varphi \in \Gamma^*$ (by 13.10 and 13.11).

   $\mathcal{I}(\exists x \varphi) = N$ iff there is $\sigma_i \in B_i$ such that $\mathcal{I}(S^{\sigma_i}_{x_i}(\varphi)) = N$ and there is no $\tau_i \in B_i$ such that $\mathcal{I}(S^{\tau_i}_{x_i}(\varphi)) = T$ iff there is $\sigma_i \in B_i$ such that $S^{\sigma_i}_{x_i}(\varphi) \notin \Gamma^*$ and $\neg S^{\sigma_i}_{x_i}(\varphi) \notin \Gamma^*$ and there is no $\tau_i \in B_i$ such that $S^{\tau_i}_{x_i}(\varphi) \in \Gamma^*$ (ind. hyp.) iff $\exists x_i \varphi \notin \Gamma^*$ and $\neg\exists x_i \varphi \notin \Gamma^*$ (by 13.10 and 13.11).

Finally, $\mathcal{I}$ is at most countable because $TERM = \bigcup_{i \in GEN-\{0\}} TERM_i$ is at most countable.

■

Proof of corollary 19.
It is an immediate consequence of lemmas 17 and 18.

■

Proof of lemma 20 (substitution of new constants for free variables).
Notation: For each $\Gamma \subseteq FORM(\mathcal{L})$ we note $\mathcal{L}^+_\Gamma$ the language obtained by adding to $\mathcal{L}$ a new constant for every free variable occurring in Γ. Let Γ^+ be the set of closed formulas of $\mathcal{L}^+_\Gamma$ obtained by systematically replacing every free variable with the corresponding new constant ($\Gamma^+ \subseteq SENT(\mathcal{L}^+_\Gamma)$).

We have to prove: if $\Gamma \subseteq FORM(\mathcal{L})$ is consistent and $\Gamma^+ \subseteq SENT(\mathcal{L}^+_\Gamma)$ has a countable strong model, then so does Γ .
Suppose that the structure $\mathcal{B} = \langle \{B_i\}_{i \in GEN-\{0\}}\ ,\ \{f^{\mathcal{B}}\}_{f \in OPER.SYM+} \rangle$ with the same signature as language $\mathcal{L}^+_\Gamma$ is a (strong) model of Γ^+ (given that Γ^+ is a set of sentences we do not need a variable assignment).
Consider the restriction of $\mathcal{B}$ to OPER.$SIM(\mathcal{L})$, eliminating the new constants that appear in $\{f^{\mathcal{B}}\}_{f \in OPER.SYM+}$, name it $\mathcal{B}^-$.
Consider the variable assignment $\mathcal{H}$, where each free variable of Γ gets as value the corresponding new constant of $\mathcal{L}^+_\Gamma$. It is clear that $\mathcal{I} = \langle \mathcal{B}^-, \mathcal{H} \rangle$ is a model of Γ.

■

Proof of theorem 21 (Henkin theorem).

Let $\Gamma \subseteq FORM(\mathcal{L})$.
(1) When FREE(Γ) is a finite set we only need corollary 19.

(2) If FREE(Γ) is not a finite set, that is, it is countable infinite, we associate to each variable of FREE(Γ) a new constant of the same genus (non appearing in $\mathcal{L}$) , c_{ij} for each $v_{ij} \in FREE(\Gamma)$ of genus i. Suppose that the variable set and the new constant set have been ordered within their genera.
Let $\mathcal{L}^+ = \mathcal{L} \cup (\bigcup_{i \in GEN-\{0\}} \{c_{i1}, c_{i2}, ...\})$.
For each $\varphi \in \Gamma$ let φ^+ be the formula obtained by replacing every free variable in φ with the corresponding new constant. Let $\Gamma^+ = \{\varphi^+ / \ \varphi \in \Gamma\}$. If we prove that Γ^+ is consistent, then, by corollary 19, Γ^+ has a countable strong model because FREE(Γ^+) $= \emptyset$ is finite. Then, by lemma 20 , Γ has a countable strong model.
Therefore we only need to prove:

Lemma: Γ^+ is consistent.

Proof of lemma.
We will prove that each finite subset of Γ^+ is consistent and then, by proposition 15, Γ^+ will be consistent.
Let Γ_0^+ be a finite subset of Γ^+. In the corresponding Γ_0 only a finite number of variables can occur free, since Γ_0 itself is finite. We know that Γ_0, as a subset of Γ must be consistent. By corollary 19, Γ_0 has a countable strong model, let $\mathcal{I} = \langle \mathcal{A}, \mathcal{H} \rangle$ be this model, ($\mathcal{I}(\varphi) = T$ for all $\varphi \in \Gamma_0$).
Consider the extension of $\mathcal{I}$, $\mathcal{I}^+ = \langle \mathcal{A}^+, \mathcal{H} \rangle$ obtained by adding the denotation of the new constants for the new constants in $\mathcal{L}^+$, such that $\mathcal{I}^+(c_i) = \mathcal{I}(x_i) = \mathcal{H}(x_i)$.
It is easy to see that $\mathcal{I}^+$ is a strong model of Γ_0^+ , that is, $\mathcal{I}^+(\varphi^+) = \mathcal{I}(\varphi) = T$ for all $\varphi^+ \in \Gamma_0^+$. Finally, by lemma 12.4, Γ_0^+ is consistent.
■

6.2 Completeness of PHLM with equality

In order to prove the strong completeness theorem for PHLM with equality we first prove the following:

Lemma 22 ***(Equals substitution [ES])***
If $\mathcal{I}$ is an interpretation, $\mathcal{I}(S^{\tau}_{x_i}\epsilon) = \mathcal{I}(S^{\rho}_{x_i}\epsilon)$ for every expression, whenever $\mathcal{I}(\tau) = \mathcal{I}(\rho)$ and x_i has the same genus as τ iff x_i has the same genus as ρ .

Proof.
It is an easy induction proof.
■

The idea of the completeness proof is like in theorem 16. We will obtain the corresponding Lindenbaum lemma 17, corollary 19, lemma 20 and Henkin theorem 21 results for PHLM with equality. Therefore, we will only need to prove the corresponding Henkin lemma for PHLM with equality.

Lemma 23 ***(Henkin)***
If Γ^* is a maximally consistent with witnesses set of formulas of $PHLM$ with equality then Γ^* has a strong model (countable when the language is countable).

Proof.
We are going to construct the model described by the formulas in Γ^*, as we did in the case of PHLM without equality but as we want our model (its structure) be normal (equality is interpret as identity), we use the passing to the suitable quotient structure technique. The process is:
The model $\mathcal{B}$ is defined as in the case of PHLM without equality, by adding the interpretation of the equality symbol:
$=^{\mathcal{B}}(\tau, \rho) = T$ iff $(\tau = \rho) \in \Gamma^*$
$=^{\mathcal{B}}(\tau, \rho) = F$ iff $(\tau = \rho) \notin \Gamma^*$ and $\neg(\tau = \rho) \in \Gamma^*$
$=^{\mathcal{B}}(\tau, \rho) = N$ iff $(\tau = \rho) \notin \Gamma^*$ and $\neg(\tau = \rho) \notin \Gamma^*$

If τ and ρ are two terms of the $PHLM$ with equality language $\mathcal{L}$, we define the following relation: $\tau \approx \rho$ iff $\tau = \rho \in \Gamma^*$

Lemma.
(1) $\approx$ *is an equivalence relation in* $TERM(\mathcal{L})$. *We will note* $[\tau]$ *to the equivalence class of the term* τ.
(2) $\approx$ *is a congruence relation.*

Proof of lemma.

1. *Reflexivity*: for any $\tau \in TERM$, $\emptyset \vdash V(\tau = \tau)$ by (RE), then $\tau = \tau \in \Gamma*$, because Γ^* is maximally consistent.
Symmetry: For any $\tau, \rho \in TERM$ such that $\tau = \rho \in \Gamma^*$ then $\rho = \tau \in \Gamma^*$: $\Gamma^* \vdash V(\tau = \tau)$ because it is a theorem of the calculus. Suppose that $\tau = \rho \in \Gamma^*$, then $\Gamma^*, \tau = \rho \vdash V(\rho = \tau)$ by (SUB), so $\Gamma^* \vdash V(\rho = \tau)$. Finally $\rho = \tau \in \Gamma^*$, since Γ^* is maximally consistent.
Transitivity: for any $\tau, \sigma, \rho \in TERM$, suppose $\tau = \rho \in \Gamma^*$ and $\rho = \sigma \in \Gamma^*$:
$\Gamma^* \vdash V(\tau = \rho)$. But $\rho = \sigma \in \Gamma^*$, so $\Gamma^*, \rho = \sigma \vdash V(\tau = \rho)$. Then , by (SUB) $\Gamma^* \vdash V(\tau = \sigma)$ and thus $\tau = \sigma \in \Gamma*$.

2. (a) For any $f \in OPER.SYM$ with RANK$(f) = \langle i_0, i_1, ..., i_n\rangle$ and $i_0 \neq 0$, if $\tau_1, ..., \tau_n$ and $\sigma_1, ..., \sigma_n$ are two term sequences such that $\tau_k \approx \sigma_k (1 \leq k \leq n)$ and τ_k , σ_k have genus i_k , then $f^{\mathcal{B}}(\tau_1, ..., \tau_n) \approx f^{\mathcal{B}}(\sigma_1, ..., \sigma_n)$: if $\tau_k = \sigma_k \in \Gamma^* (1 \leq k \leq n)$, from $\Gamma^* \vdash V(f\tau_1...\tau_n = f\tau_1...\tau_n)$ (RE) then $\Gamma^* \vdash S^{\tau_1}_{x_1} V(f\tau_1...\tau_n = fx_1...\tau_n)$ and by SUB $\Gamma^*, (\tau_1 = \sigma_1) \vdash S^{\sigma_1}_{x_1} V(f\tau_1...\tau_n = fx_1...\tau_n)$ thus $\Gamma^* \vdash V(f\tau_1...\tau_n = f\sigma_1...\tau_n)$. Similarly for τ_i for all $i = 2, ..n$. Finally $\Gamma^* \vdash V(f\tau_1...\tau_n = f\sigma_1...\sigma_n)$
Then $f\tau_1...\tau_n = f\sigma_1...\sigma_n \in \Gamma^*$, thus $f\tau_1...\tau_n \approx f\sigma_1...\sigma_n$, so $f^{\mathcal{B}}(\tau_1, ..., \tau_n) \approx f^{\mathcal{B}}(\sigma_1, ..., \sigma_n)$.
(b) For any $r \in OPER.SYM$ with RANK$(r) = \langle 0, i_1, ..., i_n\rangle$ and $i_1 \neq 0$, if $\tau_1, ..., \tau_n$ and $\sigma_1, ..., \sigma_n$ are two terms sequences such that $\tau_k \approx \sigma_k (1 \leq$

$k \leq n)$ and τ_k , σ_k have genus i_k , then $r^{\mathcal{B}}(\tau_1, ..., \tau_n) \approx r^{\mathcal{B}}(\sigma_1, ..., \sigma_n)$:
if $\tau_k = \sigma_k \in \Gamma * (1 \leq k \leq n)$, we can prove by induction and using SUB that
$\Gamma^*, Vr\tau_1...\tau_n \vdash Vr\sigma_1...\sigma_k\ \tau_{k+1}...\tau_n$ for all $k(1 \leq k \leq n)$, and that
$\Gamma^*, Vr\sigma_1...\sigma_n \vdash Vr\tau_1...\tau_k\ \sigma_{k+1}...\sigma_n$ for all $k(1 \leq k \leq n)$.
Then $\Gamma^*, Vr\tau_1...\tau_n \vdash Vr\sigma_1...\sigma_n$ and $\Gamma^*, Vr\sigma_1...\sigma_n \vdash Vr\tau_1...\tau_n$. Finally, as Γ^* is maximally consistent, we conclude that $r\tau_1...\tau_n \in \Gamma^*$ iff $r\sigma_1...\sigma_n \in \Gamma^*$. Reasoning in the same way for $\neg r\tau_1...\tau_n$ and $\neg r\sigma_1...\sigma_n$, we also obtain that $\neg r\tau_1...\tau_n \in \Gamma^*$ iff $\neg r\sigma_1...\sigma_n \in \Gamma^*$ and then $r^{\mathcal{B}}(\tau_1, ..., \tau_n) = r^{\mathcal{B}}(\sigma_1, ..., \sigma_n)$.
(c) For any $r \in OPER.SYM$ such that RANK$(r) = n$, $r^{\mathcal{B}}(\tau_1, ..., \tau_n) \approx r^{\mathcal{B}}(\sigma_1, ..., \sigma_n)$, if $\tau_1, ..., \tau_n$ and $\sigma_1, ..., \sigma_n$ are two sequences of terms of any genus such that $\tau_k \approx \sigma_k (1 \leq k \leq n)$:
If $\tau_k = \sigma_k \in \Gamma^*(1 \leq k \leq n)$ then $\sigma_k = \tau_k \in \Gamma^*$ and we prove by induction, like in (b) that $r\tau_1...\tau_n \in \Gamma^*$ iff $r\sigma_1...\sigma_n \in \Gamma^*$ and that $\neg r\tau_1...\tau_n \in \Gamma^*$ iff $\neg r\sigma_1...\sigma_n \in \Gamma^*$. Then $r^{\mathcal{B}}(\tau_1, ..., \tau_n) = r^{\mathcal{B}}(\sigma_1, ..., \sigma_n)$.■

The immediate consequence of the previous lemma is the well definition of the quotient $\mathcal{A} = \mathcal{B}/_{\approx}$ and that $\mathcal{A}$ is a normal structure with the same signature as $\mathcal{B}$ and, thus, the same signature as $\mathcal{L}$, the language of $PHLM$ with equality.

The quotient structure $\mathcal{A} = \mathcal{B}/_{\approx}$ is

$\mathcal{A} = \langle \{A_i\}_{i\epsilon GEN}, \{f^{\mathcal{A}}\}_{f\epsilon OPER.SYM} \rangle$ where
(1) $A_i = B_i/_{\approx} = \{[\tau_i]/\ \ \tau_i \in TERM_i(\mathcal{L})\}$ if $i \in GEN - \{0\}\}$.
(2) for all $f \in OPER.SYM$, $f^{\mathcal{A}} = f^{\mathcal{B}}/_{\approx}$, that is,

(2.1) if RANK$(f) = \langle i_0, i_1, ..., i_n \rangle$, $i_0 \neq$, then $f^{\mathcal{A}} : A_{i_1} \times ... \times A_{i_n} \longrightarrow A_{i_0}$ and $f^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = f^{\mathcal{B}}/_{\approx}([\tau_1], ..., [\tau_n]) = [f^{\mathcal{B}}(\tau_1, ..., \tau_n)] = [f\tau_1, ..., \tau_n]$.

(2.2) for all $r \in OPER.SYM$ such that RANK$(r) = \langle 0, i_1, ..., i_n \rangle$, $r^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = r^{\mathcal{B}}(\tau_1, ..., \tau_n)$.

(2.3) for all $r \in OPER.SYM$ such that RANK$(r) = n$, $r^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = r^{\mathcal{B}}(\tau_1, ..., \tau_n)$. In particular, we have $[\tau] =^{\mathcal{A}} [\rho]$ iff $\tau =^{\mathcal{B}} \rho$,

Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{H} \rangle$ be an interpretation, with $\mathcal{H} : \bigcup_{i \in GEN-\{0\}} \mathcal{V}_i \longrightarrow \bigcup_{i \in GEN-\{0\}} A_i$
$x_i \longrightarrow [x_i]$
Now we obtain that $\mathcal{I}$ is a strong model of Γ^* by proving:

Lemma.

1. *For every* $\tau \in TERM \quad I(\tau) = [\tau]$.

2. *For every* $\varphi \in FORM$:

$$I(\varphi) = T \text{ iff } \varphi \in \Gamma^* \ ,$$
$$I(\varphi) = F \text{ iff } \varphi \notin \Gamma^* \text{ and } \neg\varphi \in \Gamma^*$$
$$I(\varphi) = N \text{ iff } \varphi \notin \Gamma^* \text{ and } \neg\varphi \notin \Gamma^* \ .$$

Proof of lemma.

1. The proof for terms is by induction on the term length:
   $\mathcal{I}(x_i) = \mathcal{H}(x_i) = [x_i]$
   $\mathcal{I}(f\tau_1...\tau_n) = f^{\mathcal{A}}(\mathcal{I}(\tau_1), ..., \mathcal{I}(\tau_n)) = f^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = [f\tau_1, ...\tau_n]$.
   (Where f is a function symbol of type $\langle i_0, ..., i_n\rangle$ with $i_0 \neq 0$). We use the induction hypothesis on the terms $\tau_k (1 \leq k \leq n)$ of genus i_k.

2. The proof for formulas is by induction on the construction of the formulas, the number of occurrences of $\neg$, V, , ξ , $\vee$ and $\exists$ is like the one given in the case of PHLM without equality, by using the consistence results and the deduction rules. The only special case is for the atomic formulas:
   (a) If $\varphi \equiv r\tau_1...\tau_n$ with r a relation symbol of type $\langle 0, i_1, ..., i_n\rangle$ with $i_1 \neq 0$ or r an untyped relation symbol of ariety n :
   $\mathcal{I}(\varphi) = T$ iff $\mathcal{I}(r\tau_1...\tau_n) = r^{\mathcal{A}}(\mathcal{I}(\tau_1), ..., \mathcal{I}(\tau_n)) = r^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = T$ iff $r\tau_1...\tau_n \in \Gamma^*$.
   $\mathcal{I}(\varphi) = F$ iff $\mathcal{I}(r\tau_1...\tau_n) = r^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = F$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \in \Gamma^*$
   $\mathcal{I}(\varphi) = N$ iff $\mathcal{I}(r\tau_1...\tau_n) = r^{\mathcal{A}}([\tau_1], ..., [\tau_n]) = N$ iff $r\tau_1...\tau_n \notin \Gamma^*$ and $\neg r\tau_1...\tau_n \notin \Gamma^*$.
   (b) If $\varphi \equiv \tau = \rho$, $\mathcal{I}(\tau = \rho) = T$ iff $\tau = \rho \in \Gamma^*$
   $=^{\mathcal{B}}(\tau, \rho) = F$ iff $(\tau = \rho) \notin \Gamma^*$ and $\neg(\tau = \rho) \in \Gamma^*$
   $=^{\mathcal{B}}(\tau, \rho) = N$ iff $(\tau = \rho) \notin \Gamma^*$ and $\neg(\tau = \rho) \notin \Gamma^*$ by definition.■

Finally, $\mathcal{I}$ is at most countable because $TERM = \bigcup_{i \in GEN - \{0\}} TERM_i$ is at most countable.
■

6.3 Weak completeness of the calculus

Before enunciating the next theorem (weak completeness), which connects the satisfiability concept with the deductive calculus, we have to recall that we have two concepts of validity, strong and weak.
A formula φ is *valid (strongly)* if any interpretation $\mathcal{I}$ is a strong model, that is, $\mathcal{I}(\varphi) = T$, and we write $\vDash_s \varphi$ or simply $\vDash \varphi$. This was the chosen validity (satisfiability) concept for PHLM. And φ is *weakly valid* if any interpretation $\mathcal{I}$ is a weak model, that is, $\mathcal{I}(\varphi) \neq F$, and we write $\vDash_w \varphi$.

So, we can enunciate the lemma of expressing validity by using the semantical consequence $\models \equiv \models_{sw}$.

Lemma 24

1. $\models \varphi$ *iff* $\emptyset \models V\varphi$
2. $\models_w \varphi$ *iff* $\emptyset \models \varphi$

Proof.
$\models \varphi$ iff for any interpretation $\mathcal{I}$, $\mathcal{I}(\varphi) = T(\mathcal{I}(V\varphi) \neq F)$.
Now consider that $\models_w \varphi$ iff for any interpretation $\mathcal{I}(\varphi) \neq F$.
■

Lemma 25 *Let* $\varphi \in FORM$*, if* φ *is valid (strongly) then* $\vdash V\varphi$*.*

Proof.
It is a corollary of the strong completeness theorem and the previous lemma..
■

Theorem 26 ***(Weak Completeness)*** *Let* $\varphi \in FORM$*, if* φ *is weakly valid then* $\vdash \varphi$*.*

Proof.
It is a corollary of lemma 25 and rule *EVC*.
■

If we want that the *theorem concept in PHLM* be the syntactical counterpart of the semantical concept of *(strongly) valid formula*, then φ must be a theorem if and only if $\vdash V\varphi$ in the calculus.
If $\vdash \varphi$, we only have the weak validity, which is not the chosen validity for our semantics, and we can only say that φ is a *derivation* of the calculus.

7 Compactness and Löwenheim-Skolem theorems for PHLM

Theorem 27 (compactness) Γ *has a (strong) model iff every finite subset of* Γ *has a (strong) model.*

Proof.
Γ has a (strong) model iff Γ is consistent (by theorem 21 and 12.4) iff every finite subset of Γ is consistent (by lemma 15) iff every finite subset of Γ has a (strong) model (by 21 and 12.4).
■

Theorem 28 (Löwenheim-Skolem) *Let Γ be a set of heterogeneous formulas written in a countable language.*
If Γ has a (strong) model then Γ has a (strong) model at most countable.

Proof.
If Γ has a (strong) model then Γ is consistent (by 12.4). Then it has a (strong) model at most countable (by theorem 21).
$\blacksquare$

Both theorems are also valid for PHLM with equality, in this case the proof uses the results presented in the study of completeness with equality.

8 References

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