# A study of negative modalities

João Marcos

UFRN (BR) & RUB (DE)

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[Popper 1943, 1959, 1963]
I thought indeed about a system in which contradictory sentences were not 'embracing', that is, did not explode, but abandoned this system because it turned out to be *too weak*.

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**Note that**: (1) congruentiality $\succ$ extensionality $\succ$ truth-functionality
(2) typical examples of congruential logics: classical modal logics

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$wski(p) = p$

$wski(\sim\alpha) = \sim wski(\alpha)$

$wski(\alpha \vee \beta) = wski(\alpha) \vee wski(\beta)$ $\qquad jask(\varphi) = \Diamond\varphi$

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Check that $p$ and $\neg\neg p$ are atom-equivalent yet not $\sim$-equivalent in $D2$.

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More generally, every $\sim$-paraconsistent extension of $CPL^+$ *fails congruentiality* if it happens to sanction the inference $\sim(\alpha \supset \beta) \Vdash \alpha \wedge \sim\beta$.

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No (boldly) paraconsistent logic with a deductive implication $\supset$ can sanction the inference $\beta \supset \alpha \Vdash {\sim}\alpha \supset {\sim}\beta$. [Popper 1963]

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*Indeed:*

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And now for something completely different:

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Consider Kripke models for $CPL^+$ and add a unary connective $\smile$ such that:

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\bot & := & \smile(p \supset p) \\
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**Then the logics thereby defined. . .** [J.M., Log & Anal 2005]

. . . are $\smile$-reversing, $\neg$-reversing and $\frown$-reversing

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. . . generate precisely the *normal modal logics*

. . . are $\smile$-paraconsistent (and $\frown$-paracomplete)

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**The Challenge:**

To do this without the help of an implication in the language!

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*System* $\mathcal{K}$ : *CPL*$^+$ extended by

[K] $\vdash \smile(\alpha \wedge \beta) \supset (\smile\alpha \vee \smile\beta)$

[N1] If $\vdash \alpha \supset \beta$, then $\vdash \smile\beta \supset \smile\alpha$

[N2] If $\vdash \alpha$, then $\vdash \smile\alpha \supset \beta$

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|---|---|
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| [N1] | If $\vdash \alpha \supset \beta$, then $\vdash \smile\beta \supset \smile\alpha$ |
| [N2] | If $\vdash \alpha$, then $\vdash \smile\alpha \supset \beta$ |

| *System* | *Axiom* | *Frames* |
|---|---|---|
| $\mathcal{KT}$ | $\alpha \vee \smile\alpha$ | reflexive |
| $\mathcal{KB}$ | $\smile\smile\alpha \supset \alpha$ | symmetric |
| $\mathcal{K}5$ | $(\smile\alpha \wedge \smile\smile\alpha) \supset \beta$ | euclidean |
| $\mathcal{K}4$ | $(\frown\alpha \wedge \smile\frown\alpha) \supset \beta$ | transitive |
| $\mathcal{K}2$ | $(\smile\alpha \wedge \frown\smile\alpha) \supset \beta$ | dense |
| $\mathcal{K}G$ | $\smile\smile\alpha \supset \frown\frown\alpha$ | confluent |

# A parsimonious modal (interpreted) language

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**Frame**: $\mathcal{F} = \langle W, R \rangle$, where $W \neq \varnothing$ and $R \subseteq W \times W$

**Model**: $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a frame, and $V : W \times \mathcal{L} \to \{f, t\}$ respects (we use $\mathcal{M}, z \Vdash \alpha$ to denote $V(z, \alpha) = t$)

[S⊃] $\mathcal{M}, w \Vdash \varphi \supset \psi$ iff $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \Vdash \psi$

[S⌣] $\mathcal{M}, w \Vdash \smile\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that $wRv$

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[S$\supset$] $\mathcal{M}, w \Vdash \varphi \supset \psi$ iff $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \Vdash \psi$

[S$\smile$] $\mathcal{M}, w \Vdash \smile\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for some $v \in W$ such that $wRv$

**Recovering the (now standard) basic modal languages**

$\sim\alpha := \alpha \supset \smile(\alpha \supset \alpha)$ behaves as the classical negation

and

$\square\alpha := \sim\smile\alpha$ behaves as the usual (positive) modality box

# A parsimonious modal (interpreted) language

**Frame**: $\mathcal{F} = \langle W, R \rangle$, where $W \neq \varnothing$ and $R \subseteq W \times W$

**Model**: $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a frame,
and $V : W \times \mathcal{L} \to \{f, t\}$ respects (we use $\mathcal{M}, z \Vdash \alpha$ to denote $V(z, \alpha) = t$)

[S$\supset$] $\mathcal{M}, w \Vdash \varphi \supset \psi$ iff $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \Vdash \psi$

[S$\smile$] $\mathcal{M}, w \Vdash {\smile}\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that $wRv$

## Recovering the (now standard) basic modal languages

${\sim}\alpha := \alpha \supset {\smile}(\alpha \supset \alpha)$ behaves as the **classical negation**

and

$\Box\alpha := {\sim}{\smile}\alpha$ behaves as the usual **(positive) modality box**

<u>Note 0</u>. Conversely, the (paraconsistent) negation $\smile$ might be recovered through ${\sim}\Box$.

<u>Note 1</u>. It is reasonable to expect $\smile$ to be, in general, weaker than $\sim$, i.e.:

$${\sim}\alpha \models {\smile}\alpha, \quad \text{yet} \quad {\smile}\alpha \not\models {\sim}\alpha$$

<u>Note 2</u>. Our minimal language, in what follows, will be:

$$\mathcal{L}_{\wedge\vee\top\bot}, \text{ classically interpreted}$$

# Unary modal operators

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The classical case:

| | $\curlyvee$ | $+$ | $-$ | $\curlywedge$ |
|---|---|---|---|---|
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[PM2.2+] $+(\varphi \lor \psi) \succ +\varphi \lor +\psi$

**type [+]**

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| [PT+] | $\succ +\top$ | | |

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$[\text{PM2.2+}]\quad +(\varphi \vee \psi) \succ +\varphi \vee +\psi$

**full type <+>**

$[\text{PB+}]\quad +\bot \succ$

Case of [rev]:

$[\text{DM1.1}-]\quad -(\varphi \vee \psi) \succ -\varphi \wedge -\psi$

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**full type <+>**

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$[\mathrm{PT}-]\quad \succ -\bot$

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**full type <->**

$[\mathrm{PB}-]\quad -\top \succ$

**Negations?**

[J.M., J Appl Log 2005]

$[\![\textit{falsificatio}]\!]\quad \forall k \exists p\ \#^k p \not\succ \#^{k+1} p$

$[\![\textit{verificatio}]\!]\quad \forall k \exists p\ \#^{k+1} p \not\succ \#^k p$

# [Boxes] and <Diamonds>

Let $\sim$ represent **classical negation**. Then:

$$[+]{\sim}\alpha \equiv {\sim}{<}{+}{>}\alpha \qquad {<}{+}{>}{\sim}\alpha \equiv {\sim}[+]\alpha$$

$$[-]{\sim}\alpha \equiv {\sim}{<}{-}{>}\alpha \qquad {<}{-}{>}{\sim}\alpha \equiv {\sim}[-]\alpha$$

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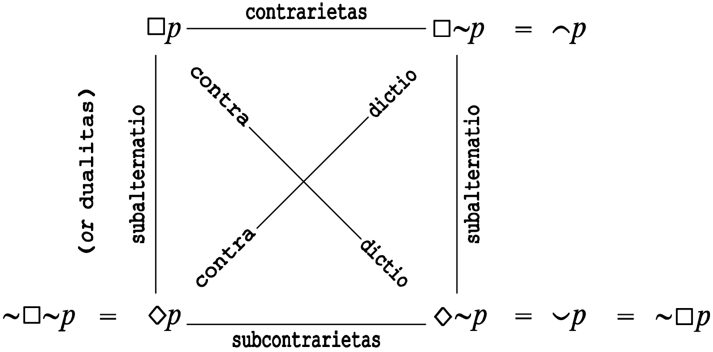


**Figure:** *Square of Modalities* (som)

[J.M., Log & Anal 2005]

# Non-classical behavior and derivability adjustment

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The two sides of **negation**:

$[\![\#\text{-explosion}]\!]$ $\quad p, \#p \succ q$ $\qquad\qquad$ $[\![\#\text{-implosion}]\!]$ $\quad q \succ \#p, p$

# Non-classical behavior and derivability adjustment

The two sides of **negation**, and their *failures*:

$[\![\#\text{-explosion}]\!]$ $p, \#p \nprec q$ paraconsistency

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| | paraconsistency | | paracompleteness |

Some (strong) **gentler** versions:

| | | | | | |
|---|---|---|---|---|---|
| [C1#] | $\circledast p, p, \# p \succ$ | [C2#] | $\succ p, \circledast p$ | [C3#] | $\succ \# p, \circledast p$ |
| [D1#] | $\succ \# p, p, \circledast p$ | [D2#] | $\circledast p, p \succ$ | [D3#] | $\circledast p, \# p \succ$ |

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**Enriching the object language through adjustment connectives**

The [C$n$#] clauses (strongly) internalize the 'consistency assumption', and the [D$n$#] clauses (strongly) internalize the 'determinedness assumption'.

This defines (strong versions of) the so-called **LFI**s and **LFU**s.

# Modal semantics: *it ain't necessarily so!*

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[S$\smile$] $\mathcal{M}, w \Vdash {\smile}\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that $wRv$

[S$\frown$] $\mathcal{M}, w \nVdash {\frown}\varphi$ iff $\mathcal{M}, v \Vdash \varphi$ for <u>some</u> $v \in W$ such that $wRv$

equivalently:

$\mathcal{M}, w \Vdash {\frown}\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>every</u> $v \in W$ such that $wRv$

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*It should be noted that*, for <u>non-degenerate</u> classes of frames:

$\smile$ is a **paraconsistent negation**,
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Consider also the following *adjustment connectives*:

[SC$\ominus$] $\mathcal{M}, w \Vdash \ominus\varphi$ iff $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \nVdash \smile\varphi$

[SD$\oslash$] $\mathcal{M}, w \nVdash \oslash\varphi$ iff $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \frown\varphi$

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[SD$\ominus$] $\mathcal{M}, w \nVdash \ominus\varphi$ iff $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash {\frown}\varphi$

*It should be noted that:*

- $\ominus$ **expresses $\smile$-consistency**,
  and allows for $\smile$-**explosiveness** to be recovered
- $\ominus$ **expresses $\frown$-determinedness**,
  and allows for $\frown$-**implosiveness** to be recovered

# Classical negation: interactions, and definability

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Intuition: $\boxed{\sim = \frown + \smile}$

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## How does classical negation relate to the non-classical ones?

Recall that: $[-]{\sim}\alpha \;\equiv\; {\sim}\langle-\rangle\alpha$ $\qquad$ $\langle-\rangle{\sim}\alpha \;\equiv\; {\sim}[-]\alpha$

In other words: $\frown{\sim}\alpha \;\equiv\; {\sim}\smile\alpha$ $\qquad$ $\smile{\sim}\alpha \;\equiv\; {\sim}\frown\alpha$

*Moreover, in some situations:* $\frown\alpha \succ {\sim}\alpha$ $\qquad$ ${\sim}\alpha \succ \smile\alpha$

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## What if classical negation is *not* taken as a primitive connective?

To investigate: In which situations is it even *definable* in $\mathcal{L}_{\wedge\vee\top\bot\smile\frown\ominus\circleddash}$?

(we'll answer this later on!)

# A sequent calculus for PK

# A sequent calculus for $\mathrm{PK}$

A sequent calculus for the weakest normal modal logic over $\mathcal{L}_{\wedge\vee\top\bot\smile\frown\ominus\circleddash}$:

[A. Dodó & J.M., ENTCS 2014]

$$[id]\quad \frac{}{\varphi \Rightarrow \varphi} \qquad\qquad [cut]\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

$$[W{\Rightarrow}]\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad\qquad [{\Rightarrow}W]\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

$$[\bot{\Rightarrow}]\quad \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad\qquad [{\Rightarrow}\top]\quad \frac{}{\Gamma \Rightarrow \top, \Delta}$$

$$[\wedge{\Rightarrow}]\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad\qquad [{\Rightarrow}\wedge]\quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

$$[\vee{\Rightarrow}]\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad\qquad [{\Rightarrow}\vee]\quad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

$$[\smile{\Rightarrow}]\quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\frown\Delta, \smile\varphi \Rightarrow \smile\Gamma} \qquad\qquad [{\Rightarrow}\frown]\quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\frown\Delta \Rightarrow \frown\varphi, \smile\Gamma}$$

$$[\ominus{\Rightarrow}]\quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \smile\varphi, \Delta}{\Gamma, \ominus\varphi \Rightarrow \Delta} \qquad\qquad [{\Rightarrow}\ominus]\quad \frac{\Gamma, \varphi, \smile\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \ominus\varphi, \Delta}$$

$$[\circleddash{\Rightarrow}]\quad \frac{\Gamma \Rightarrow \varphi, \frown\varphi, \Delta}{\Gamma, \circleddash\varphi \Rightarrow \Delta} \qquad\qquad [{\Rightarrow}\circleddash]\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \frown\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \circleddash\varphi, \Delta}$$

# The technology of Basic Sequents

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A **basic rule**: *main* sequent + *context* sequent [O. Lahav & A. Avron 2013]

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*Examples:*

$$[\wedge\Rightarrow] \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad [\Rightarrow\wedge] \quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

are described as:

$$[\wedge\Rightarrow] \quad \langle p_1, p_2 \Rightarrow ; \pi_0 \rangle \;/\; p_1 \wedge p_2 \Rightarrow \qquad [\Rightarrow\wedge] \quad \langle \Rightarrow p_1 \;;\; \pi_0 \rangle, \langle \Rightarrow p_2 \;;\; \pi_0 \rangle \;/\; \Rightarrow p_1 \wedge p_2$$

where $\pi_0 = \{\langle q_1 \Rightarrow \;;\; q_1 \Rightarrow \rangle, \langle \Rightarrow q_1 \;;\; \Rightarrow q_1 \rangle\}$.

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*while*

$$[\smile\Rightarrow] \quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\frown\Delta, \smile\varphi \Rightarrow \smile\Gamma}$$

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The **main sequent** is made to match an appropriate *semantic condition*.
For instance, $[\smile\Rightarrow]$ induces:

| |
|---|
| "if $\mathcal{M}, v \Vdash\, \Rightarrow \alpha$ for every world $v$ such that $wRv$, then $\mathcal{M}, w \Vdash \smile\alpha \Rightarrow$" |

and the **context sequent** is also made to match a *semantic condition*.
For instance, $\pi_1$ induces:

| |
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| "if $wRv$ then $\mathcal{M}, w \Vdash\, \Rightarrow \smile\beta$ whenever $\mathcal{M}, v \Vdash \beta \Rightarrow$" |

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Together, these correspond precisely to:

$[S\smile]$ $\quad \mathcal{M}, w \Vdash \smile\varphi \quad$ iff $\quad \mathcal{M}, v \nVdash \varphi$ for some $v \in W$ such that $wRv$

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For our convenience, we rewrite this as:

$[\mathbf{F}\smile]$ $\quad$ if $\mathbf{T}_v(\varphi)$ for every $v \in W$ such that $wRv$, then $\mathbf{F}_w(\smile\varphi)$

$[\mathbf{T}\smile]$ $\quad$ if $\mathbf{F}_v(\varphi)$ for some $v \in W$ such that $wRv$, then $\mathbf{T}_w(\smile\varphi)$

where we take '$\mathbf{T}_u(\alpha)$' as abbreviating '$V(u, \alpha) = t$', and '$\mathbf{F}_u(\alpha)$' as abbreviating '$V(u, \alpha) = f$'.

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Say that $w, v \in W$ **agree with respect to the formula** $\alpha$**, according to** $V$,
if either ($\mathbf{T}_w(\alpha)$ and $\mathbf{T}_v(\alpha)$) or ($\mathbf{F}_w(\alpha)$ and $\mathbf{F}_v(\alpha)$).

Call $\mathcal{M}$ a **differentiated** model
if $w = v$ whenever $w$ and $v$ agree with respect to every $\alpha \in \mathcal{L}$, according to $V$.

Call $\mathcal{M}$ a **strengthened** model iff $wRv$

if ($\mathbf{T}_v(\alpha)$ implies $\mathbf{F}_w(\frown\alpha)$) and ($\mathbf{F}_v(\alpha)$ implies $\mathbf{T}_w(\smile\alpha)$), for every $\alpha \in \mathcal{L}$.

# The technology of Basic Sequents

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## Adequacy Theorem.

(corollary of [O. Lahav & A. Avron 2013])

PK is *sound* and *complete* with respect to any class of Kripke models that: (*i*) contains only models that satisfy all the appropriate $[\mathbf{T}\#]$ and $[\mathbf{F}\#]$ conditions; and (*ii*) contains all strengthened differentiated models that satisfy all the appropriate $[\mathbf{T}\#]$ and $[\mathbf{F}\#]$ conditions.

# Cut-elimination and analyticity, almost for free

check 'Sequent systems for negative modalities', O. Lahav, J.M., and Y. Zohar, Log Univ 2017

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A strategy in <u>two steps</u>:

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**Step 1.** Present an adequate semantics for the cut-free fragment of $\mathrm{PK}$.

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We now build models with **quasi valuations**

$QV : W \times \mathcal{L} \to \{\{f\}, \{t\}, \{f, t\}\}$ such that:

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$\boxed{\text{Note that these are in principle } \textbf{non-deterministic}!}$

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**Step 2.** Show that the existence of a **countermodel in the form of a strengthened differentiated quasi model** implies the existence of an ordinary countermodel.

Let an **instance** of a quasi model $\mathcal{QM} = \langle\langle W, R\rangle, QV\rangle$ be any model $\mathcal{M} = \langle\langle W, R'\rangle, V\rangle$ such that $\mathbf{X}^Q_w(\varphi)$ whenever $\mathbf{X}_w(\varphi)$, for every $\mathbf{X} \in \{\mathbf{T}, \mathbf{F}\}$, every $w \in W$ and every $\varphi \in \mathcal{L}$.

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**Theorem.**

Every quasi model has an instance.

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Consider the **well-founded relation** $\ll$ on $\mathcal{L}$ such that $\alpha \ll \beta$ iff either:
(*i*) $\alpha$ is a proper subformula of $\beta$, or
(*ii*) $\alpha = \smile\gamma$ and $\beta = \circledsmile\gamma$ for some $\gamma \in \mathcal{L}$, or
(*iii*) $\alpha = \frown\gamma$ and $\beta = \circledfrown\gamma$ for some $\gamma \in \mathcal{L}$.

Since the class of all quasi models contains the strengthened differentiated quasi models, it follows that:

## Corollary

$\mathrm{PK}$ enjoys cut-admissibility.

## Corollary

$\mathrm{PK}$ is $\ll$-analytic:
If a sequent *s* is derivable from a set *S* of sequents in $\mathrm{PK}$, then there is a derivation of *s* from *S* such that every formula $\varphi$ that occurs in the derivation satisfies $\varphi \ll \psi$ for some $\psi$ in $S \cup s$.

# Seriality, Reflexivity, Functionality, Symmetry

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| sequent system | : | frames | : | some distinguishing **features** |
|---|---|---|---|---|
| $PKD$ | : | serial | : | $\frown p \models \smile p$ |
| $PKT$ | : | reflexive | : | $p, \frown p \models q$ and $q \models \smile p, p$ |
| $PKF$ | : | total functional | : | $\smile p \equiv \frown p$ |
| $PKB$ | : | symmetric | : | $\smile\smile p \models p$ and $p \models \frown\frown p$ |

$$[\mathbf{D}] \quad \frac{\Gamma \Rightarrow \Delta}{\frown\Delta \Rightarrow \smile\Gamma}$$

$$[\mathbf{T}_1] \quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \smile\varphi, \Delta} \qquad [\mathbf{T}_2] \quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \frown\varphi \Rightarrow \Delta}$$

$$[\mathbf{Fun}] \quad \frac{\Gamma \Rightarrow \Delta}{\smile\Delta \Rightarrow \smile\Gamma}$$

$$[\mathbf{B}_1] \quad \frac{\Gamma, \smile\Gamma', \varphi \Rightarrow \Delta, \frown\Delta'}{\frown\Delta, \Delta' \Rightarrow \frown\varphi, \smile\Gamma, \Gamma'} \qquad [\mathbf{B}_2] \quad \frac{\Gamma, \smile\Gamma' \Rightarrow \varphi, \Delta, \frown\Delta'}{\frown\Delta, \Delta', \smile\varphi \Rightarrow \smile\Gamma, \Gamma'}$$

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Links to the full story: [AiML 2016] and [Log. Univ. 2017].

# On the definability of classical negation

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## When $\sim$ is not a primitive connective

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## When $\sim$ is not a primitive connective

1. Classical negation *is definable* in the logics:
   $PKT$-$\{\smile, \circledsmile\}$, (set $\sim\varphi := \frown\varphi \vee \circledfrown\varphi$)
   $PKT$-$\{\frown, \circledfrown\}$, (set $\sim\varphi := \smile\varphi \wedge \circledsmile\varphi$)
   $PKD$, and $PKF$. (set $\sim\varphi := (\frown\varphi \wedge \circledsmile\varphi) \vee \circledfrown\varphi$)

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2. Classical negation *is not definable* in the logics:
   $PK$, $PKB$,
   $PKT$-$\{\circledsmile, \circledfrown\}$,
   $PKD$-$\{\circledsmile\}$, $PKF$-$\{\circledsmile\}$,
   $PKD$-$\{\circledfrown\}$, and $PKF$-$\{\circledfrown\}$.

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More on the study of *negative modalities*:

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More on the study of *negative modalities*:

- Studying **properties of negation** through other classes of frames
  *Examples:*

| | | |
|---|---|---|
| **Church-Rosser** | : | $\smallsmile\smallsmile p \models \smallfrown\smallfrown p$ |
| **transitive** | : | $\smallsmile p, \smallsmile(\smallsmile p) \models q$ and $q \models \smallfrown(\smallfrown p), \smallfrown p$ |
| **euclidean** | : | $\smallfrown p, \smallsmile(\smallfrown p) \models q$ and $q \models \smallfrown(\smallsmile p), \smallsmile p$ |

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  | **euclidean** | : | $\frown p, \smile(\frown p) \models q$ and $q \models \frown(\smile p), \smile p$ |

- Studying combinations of **negations of the same type**.
  *Examples:* add the **backward-looking** modalities

  [S$\smile^{-1}$] $\mathcal{M}, w \Vdash \smile^{-1}\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that $wR^{-1}v$

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  *Note* the validity in *PK* of **pure** consecutions such as:
  $\smile^{-1}\smile p \models p$ and $\smile\smile^{-1} p \models p$ (as well as $p \models \frown^{-1}\frown p$ and $p \models \frown\frown^{-1} p$)
  *and* the validity in *PKB* of **mixed** consecutions such as
  $\frown^{-1}\smile p \models p$ and $\frown\smile^{-1} p \models p$ (as well as $p \models \smile^{-1}\frown p$ and $p \models \smile\frown^{-1} p$)

# *This is possibly not the end!*

More on the study of *negative modalities*:

- Studying **properties of negation** through other classes of frames
  *Examples:*

  | | | |
  |---|---|---|
  | **Church-Rosser** | : | $\smile\smile p \models \frown\frown p$ |
  | **transitive** | : | $\smile p, \smile(\smile p) \models q$ and $q \models \frown(\frown p), \frown p$ |
  | **euclidean** | : | $\frown p, \smile(\frown p) \models q$ and $q \models \frown(\smile p), \smile p$ |

- Studying combinations of **negations of the same type**.
  *Examples:* add the **backward-looking** modalities
  [S$\smile^{-1}$] $\quad \mathcal{M}, w \Vdash \smile^{-1}\varphi \quad$ iff $\quad \mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that $wR^{-1}v$
  [S$\frown^{-1}$] $\quad \mathcal{M}, w \nVdash \frown^{-1}\varphi \quad$ iff $\quad \mathcal{M}, v \Vdash \varphi$ for <u>some</u> $v \in W$ such that $wR^{-1}v$
  *Note* the validity in *PK* of **pure** consecutions such as:
  $\smile^{-1}\smile p \models p$ and $\smile\smile^{-1} p \models p$ (as well as $p \models \frown^{-1}\frown p$ and $p \models \frown\frown^{-1} p$)
  *and* the validity in *PKB* of **mixed** consecutions such as
  $\frown^{-1}\smile p \models p$ and $\frown\smile^{-1} p \models p$ (as well as $p \models \smile^{-1}\frown p$ and $p \models \smile\frown^{-1} p$)

- **Hybridizing** and **higher-orderfying** this study!

# Coda: On the true meaning of diamonds

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