A characterization of Quantified Hybrid Logic with Non-constant Domains

Antonia Huertas
mhuertass@uoc.edu
Open University of Catalonia, Barcelona, Spain.
http:// www.uoc.edu*

November 14, 2005

Abstract

Hybrid Logics are Modal Logics where formulas can refer to states. Hybrid Logic expressivity allow to name a state and to assert that a formula is true at an specific state, which is not possible in standard Modal Logic.

In particular Quantified Hybrid Logic can be seen as an extension of Quantified Modal Logic.

Recent results on Quantified Hybrid Logic present an "orthodox" system that show to have a better behaviour that the corresponding quantified modal system, but this system is only dealing with constant domains, that is, the domain of first order individuals is the same for all the states.In this paper we deal with models of Quantified Hybrid Logic with non constant domains and explore them, this provide a more expressive logic that this one of a unique domain. We show in detail a Quantified Hybrid Logic with non-constant domains and we provide a syntactic characterization in terms of Heterogeneous Partial Logic.

*This work has been partialy supported by the Spanish Ministerio de Ciencia y Tecnología (proyecto BFF 2003-08998-C03-02)

Contents

1 Introduction.

Hybrid languages are extensions of modal logics which use formulas to refer to worlds. Introduced by Arthur Prior in 1967, together with temporal logics were used in the analysis of time. Recent works of Patrick Blackburn, Carlos Areces and Maarten Marx [1] has focused on a constrained propositional hybrid system called $\mathcal{H}(@, \downarrow)$. This logic will be our starting point.
In $\mathcal{H}(@, \downarrow)$ we ad tools to a propositional modal logic to name states and to assert that a formula is true at a named state:

· *Nominals*: propositional constants that are true at one state in any model. A nominal n name the only state where it is true.

· *State variables*: a set of variables to refer to states.

· *Satisfaction operator* $@_s$ (where s is a nominal or a state variable): $@_n\varphi$ asserts that φ is true at the state named by s.

· *Binder operator* $\downarrow w.$ (where w a state variable): $\downarrow w.\varphi$ binds all occur. of the state variable named by w to the current state.

2 Quantified Hybrid Logic $\mathcal{QH}(@, \downarrow)$

2.1 Quantified Hybrid Language

The initial idea is to define a quantified hybrid logic expanding $\mathcal{H}(@, \downarrow)$ with the usual first order elements.

A *quantified hybrid language* $\mathcal{L}$ is a set of formal symbols $ALPH(\mathcal{L}) \cup EXPR(\mathcal{L})$ -alphabet and expressions-, where:

- $ALPH(\mathcal{L}) = NOM \cup CON \cup FUNC \cup PRED \cup \mathcal{VAR} \cup \{\forall, \neg, \rightarrow, \Box, @, \downarrow\}$ with
  $NOM = \{n_j /_j \in \omega\}$ an infinite numerable set of state constants (*nominals)*
  $CON = \{c_j /_j \in \omega\}$ an infinite numerable set of first order constants
  $FUNC$ is the set of first order functional symbols
  $PRED$ is the set of first order predicate symbols
  $\mathcal{VAR} = FVAR \cup SVAR$ is the set of variables. $FVAR$ the first order variables and $SVAR$ the state variables, each one is an infinite numerable set of *individual variables symbols.*
  $FVAR = \{x_j /_j \in \omega\}$ and $SVAR = \{s_j /_j \in \omega\}$.
  $\forall$ is the usual first order *universal quantifier* symbol.
  $\Box$ is the modal operator symbol of *necessity* (box).
  $@$ the hybrid *satisfaction operator* and $\downarrow$ the *binder operator* .

- $EXPR(\mathcal{L}) = TERM(\mathcal{L}) \cup FORM(\mathcal{L})$ with

  · $TERM(\mathcal{L})$ is the set of *terms*, recursively defined as it is usual in first order logic:
  CON and $FVAR$ are terms
  If $f \in FUNC$ and $\{t_{1,} \ldots, t_n\} \subset TERM(\mathcal{L})$ then $ft_1 \cdots t_n \in TERM(\mathcal{L})$

  · $FORM(\mathcal{L})$ is the set of *formulas* of $\mathcal{L}$, recursively defined as usually:
  Atomic formulas: NOM and $SVAR$ are atomic formulas. If $P \in PRED$ and $\{t_{1,} \ldots, t_n\} \subset TERM(\mathcal{L})$ then $Pt_1 \cdots t_n$ is an atomic formula
  Complex formulas: if φ, ψ are formulas and $x \in FVAR$, $w \in SVAR$ and $n \in NOM \cup SVAR$ then the following are formulas

$\neg\varphi \mid \varphi \rightarrow \psi \mid \forall x\varphi \mid \Box\varphi \mid @_n\varphi \mid \downarrow w.\varphi$
If we have an equality symbol, then $t_1 = t_2$ is a formula

We will come back here later because the definition of something in this definition of $TERM(\mathcal{L})$ will be revised. But now we have to deal with the real big problem, the definition of semantics.

2.2 Quantified Hybrid Semantics

Ruth Barcan was the principal pioneer in quantified modal logic. She added quantification over individual variables to some Lewis propositional modal systems. But soon technical and philosophical objections arose [**7**]. To mix first-order logic and modal logic was not simple at all. The biggest problems appeared at the semantical level, where there are different difficult decisions concerning the semantics of quantifiers, identity, terms and predicates. Most of the choices can be made independently and different systems can be defined [**3,5,8**].
Quantified modal semantics is not standard. As a consequence there are different ways to extend propositional hybrid logic. We are going to have to make decisions in the process of extension, let we see them in detail.

Semantics of Quantified Hybrid Logic: at least three decisions to make

Quantified hybrid semantics inherite the conflictive behavior of quantified modal semantics. To simplify we focus on the three possible most significative decisions to take when defining a quantified hybrid model:

1. *Constant domain*: there is a single domain of quantification for all the states, or
   State-relative domain: there is a specific domain of quantification for every state

2. *Rigid terms*: the interpretation of every term at every state is an unique individual of the domain, or
   Non-rigid terms: the interpretation of a term depend on the state where we are evaluating. A term can be interpret as different individuals for different states.

3. *Classical interpretation*: only true and false truth values, or
   Partial interpretation: there are formulas that are nor true nor false.

Which are the best choices? The history of Quantified Modal Logic tells us that no satisfactory system can be found. What happen in Quantified Hybrid Logic? The purpose of this article is investigate the situation, and show that the richer language of quantified hybrid logic improves it.

Constant or state relative domain?

A *modal frame* is an ordered pair $\mathcal{F} = \langle \mathbf{W}, \mathbf{R} \rangle$ with $\mathbf{W}$ the non-empty set, of *states or worlds*, and $\mathbf{R}$ a binary relation on $\mathbf{W}$, named the *accessibility relation between states.*

A *skeleton* is an ordered quadruple $\mathcal{S} = \langle \mathbf{D}, \mathbf{W}, \mathbf{R}, \mathbf{Q} \rangle$ where $\langle \mathbf{W}, \mathbf{R} \rangle$ is a frame, $\mathbf{D}$ is a non-empty set, the *universe of individuals*, and $\mathbf{Q}$ is a function which assigns a non-empty subset $\mathbf{Q}(\mathbf{s})$ of $\mathbf{D}$ (named $\mathbf{D}_s$) to each state $\mathbf{s} \in \mathbf{W}$. $\mathbf{D}_s$ is named the $\mathbf{s}$ *state's domain of individuals* ($\mathbf{D}_s \neq \emptyset$ for each $\mathbf{s} \in \mathbf{W}$).

A *quantified hybrid structure* is an ordered set $\mathcal{A} = \langle \mathbf{D}, \mathbf{W}, \mathbf{R}, \mathbf{Q}, \mathbf{I}_{nom}, \mathbf{I}_{con}, \{\mathbf{I_s}\}_{\mathbf{s} \in \mathbf{W}} \rangle$ where $\langle \mathbf{D}, \mathbf{W}, \mathbf{R}, \mathbf{Q} \rangle$ is a skeleton and
$\mathbf{I}_{nom}(n) = \mathbf{n} \in \mathbf{W}$ for each $n \in NOM$
$\mathbf{I}_{con}(c) = \mathbf{c} \in \mathbf{D}$ for each $c \in CON$

1st choice:

- To have *constant domain* if $\mathbf{D}_s = \mathbf{D}$ for all $\mathbf{s} \in \mathbf{W}$, and $\langle \mathbf{D}, \mathbf{I}_s \rangle$ is an ordinary first-order model
- To have *relative state domains* if we have different $\mathbf{D}_s$ for all $\mathbf{s} \in \mathbf{W}$, and $\langle \mathbf{D_s}, \mathbf{I_s} \rangle$ is an ordinary first-order model for each $\mathbf{s} \in \mathbf{W}$

Rigid or non-rigid terms?

We use special two-sorted assignments to interpret formulas with free variables. A quantified hybrid assignment is a function h from $SVAR \cup FVAR$ to $\mathbf{W} \cup \mathbf{D}$ sending state variables to elements of $\mathbf{W}$ and first order variables to elements of $\mathbf{D}$.
Observe that a variable x is assigned to an individual of the universe $\mathbf{D}$, and then, whether $h(x)$ belongs to a determinate state‘s universe of individuals $\mathbf{D}_s$ or not only depends on the skeleton of the structure.

An *interpretation* over the structure $\mathcal{A}$ *at a state* $\mathbf{s} \in \mathbf{W}$ is a triple $\mathcal{I}_s = \langle \mathcal{A}, h, \mathbf{s} \rangle$ **,** where h is a variable assignment.
The interpretation of the terms of the language $\mathcal{L}$ with the interpretation is defined by recursion:

1. Atomic terms (the usual first order definition)
   Constants are interpreted as "names: $\bar{c} = \mathcal{I}_s(c) = \mathbf{I}_{con}(c) \in \mathbf{D}$
   Variables are also like "names": $\bar{x}_i = \mathcal{I}_s(x_i) = h(x_i) \in \mathbf{D}$

2. **2nd choice:** (two possibilities to define complex terms)

   - *Rigid terms*: the interpretation of a term $\bar{t} = \mathcal{I}_s(t) = I_s(t) \in \mathbf{D}$ is the same individual for all $\mathbf{s} \in W$
   - *Non-rigid terms*: $\bar{t} = \mathcal{I}_s(t) = \mathbf{I}_s(t) \in \mathbf{D}$ is a different individual for every $\mathbf{s} \in \mathbf{W}$. If we have relative-state domain, $I_s(t) \in \mathbf{D}_s$.

Considerations for the first and second decision

The condition of constant domains is not natural for quantified modal (hybrid) semantics when using non rigid terms. It is only attractive because its similar to the first order semantics and quantification. The natural and desirable modal behavior is to have state-relative domains. And then, at a state $\mathbf{s} \in \mathbf{W}$ quantify over the set of individuals actually existent at this state. Think of the expression "everything have a propriety P at the state **s**" [**5,8**]
Non-rigid terms (also called local) seems to be more desirable in a modal context because they allow to have an intensional interpretation of sentences like "the men who lives next door". On the other way the interpretation of constants and first order variables could be rigid, as they can be seeing as "names" for an unique object [**2,6**].

Classical or partial interpretation?

Suppose we have state-relative domains, now still we have the following desition:

3rd choice:

- *Classical interpretation:* two values *true* T and *false* F. Formulas are true or false

- *Partial three values* T, F and N (value *null*) of a partial three-valued logic: N is the null value assigned to a formula $\varphi(x)$ at a state $\mathbf{s}$ when the interpretation of x does not belong to the domain of individuals of the state $\mathbf{s}$ (also called truth-value modal gaps) [**3**,**5**].

Partial connectives and quantifiers

In [**3**] the suitable partial connectives for the "truth-value modal gaps" where presented. We start considering the same ones for our hybrid meaningless. The partial connectives are the usual Bočvar connectives (conserving the classical ones and assigning "null" to "null":

α	$\neg\alpha$
T	F
F	T
N	N

	$\alpha \wedge \beta$			$\alpha \vee \beta$			$\alpha \rightarrow \beta$		
$\alpha \backslash \beta$	T	F	N	T	F	N	T	F	N
T	T	F	N	T	T	N	T	F	N
F	F	F	N	T	F	N	T	T	N
N	N	N	N	N	N	N	N	N	N

The new partial connectives of verification (V) and defalsification (ξ) needed to have a functionally complete set of connectives for the *truth-values set* $\{T$, F, $N\}$ are:

α	$V\alpha$	$\xi\alpha$
T	T	T
F	F	N
N	F	T

Tables of the partial quantifiers $\forall$ and $\exists$ are defined bearing in mind the interpretation of quantified formulas in Quantified Hybrid Logic

$$\mathcal{I}(\forall x\varphi) = \begin{cases} T \;\Leftrightarrow\; \{\mathbf{x} \in \mathbf{D} \;/\; \mathcal{I}_x^{\mathbf{x}}(\varphi) = T\} = \mathbf{D} \\ F \;\Leftrightarrow\; \{\mathbf{x} \in \mathbf{D} / \; \mathcal{I}_x^{\mathbf{x}}(\varphi) = F\} \neq \emptyset \\ N \;\Leftrightarrow\; \{\mathbf{x} \in \mathbf{D} \;/\; \mathcal{I}_x^{\mathbf{x}}(\varphi) = N\} \neq \emptyset \text{ and } \; \{\mathbf{x} \in \mathbf{D} / \; \mathcal{I}_x^{\mathbf{x}}(\varphi) = F\} = \emptyset \end{cases}$$

$$\mathcal{I}(\exists x\varphi) = \mathcal{I}(\neg\forall x\neg\varphi)$$

Semantics of $QH(@, \downarrow)$: interpretation of terms and formulas

After making our three decisions we take the *truth-values set* $\{T$, F, $N\}$ and we consider only state-relative domain and non-rigid terms.

Syntax of local terms

As a consequence of our decisions the definition for complex terms syntax must be "local" and then we have to redefine the set $TERM(\mathcal{L})$ in 2.1:

- $CON \subset TERM(\mathcal{L}), FVAR \subset TERM(\mathcal{L})$
- For every $s \in NOM \cup SVAR$: if $f \in FUNC$ and $\{t_{1,} \ldots, t_n\} \subset TERM(\mathcal{L})$ then $@_s ft_1 \ldots t_k \in TERM(\mathcal{L})$

Interpretation of terms

- Atomic rigid terms
  $\bar{x} = h(x) \in D$
  $\bar{c} = \mathbf{I}_{con}(c) \in D$
- Complex non-rigid terms
  $\overline{@_s t} = \mathbf{I}_w(t) \in D_w$ with $w = \mathbf{I}_{con}(s)$ or $w = h(s)$

Interpretation of formulas:

- $\mathcal{I}_s(P\epsilon_1...\epsilon_k) = T \Leftrightarrow \mathcal{I}_s(\epsilon_i) \in D_s$ for all $i \in \{1, ..., k\}$ and $\langle \mathcal{I}_s(\epsilon_1), ..., \mathcal{I}_s(\epsilon_k)\rangle \in I_s(P)$
  $\mathcal{I}_s(P\epsilon_1...\epsilon_k) = F \Leftrightarrow \mathcal{I}_s(\epsilon_i) \in D_s$ for all $i \in \{1, ..., k\}$ and $\langle \mathcal{I}_s(\epsilon_1), ..., \mathcal{I}_s(\epsilon_k)\rangle \notin I_s(P)$
  $\mathcal{I}_s(P\epsilon_1...\epsilon_k) = N \Leftrightarrow$ for some $i \in \{1, ..., k\},\ \mathcal{I}_s(\epsilon_i) \notin D_s$ (only possible if $\epsilon_i \in CON$ or $\epsilon_i \in FVAR$)
- $\mathcal{I}_s(t_1 = t_2) = T \Leftrightarrow \mathbf{I}_s(t_1) = \mathbf{I}_s(t_2)$
  $\mathcal{I}_s(t_1 = t_2) = F \Leftrightarrow \mathbf{I}_s(t_1) \neq \mathbf{I}_s(t_2)$
  $\mathcal{I}_s(t_1 = t_2) = N$ never
- $\mathcal{I}_s(n) = T \Leftrightarrow \mathbf{I}_{nom}(n) = \mathbf{s}$
  $\mathcal{I}_s(n) = F$ otherwise ($\mathcal{I}_s(n) = N$ never)
- $\mathcal{I}_s(w) = T \Leftrightarrow h(w) = \mathbf{s}$
  $\mathcal{I}_s(w) = F$ otherwise ($\mathcal{I}_s(n) = N$ never)
  Observe that in the case of Fixed domain, the value N is never taken. Thus, only with non-fixed domain we have partiality.
- $\mathcal{I}_s(\neg\varphi) = T \Leftrightarrow \mathcal{I}_s(\varphi) = F$
  $\mathcal{I}_s(\neg\varphi) = F \Leftrightarrow \mathcal{I}_s(\varphi) = T$
  $\mathcal{I}_s(\neg\varphi) = N \Leftrightarrow \mathcal{I}_s(\varphi) = N$

- $\mathcal{I}_s(\varphi \to \psi) = T \Leftrightarrow \mathcal{I}_s(\varphi) \neq N,\ \mathcal{I}_s(\psi) \neq N$ and $\mathcal{I}_s(\varphi) = F$ or $\mathcal{I}_s(\psi) = T$
  $\mathcal{I}_s(\varphi \to \psi) = F \Leftrightarrow \mathcal{I}_s(\varphi) = T$ and $\mathcal{I}_s(\psi) = F$
  $\mathcal{I}_s(\varphi \to \psi) = N \Leftrightarrow \mathcal{I}_s(\varphi) = N$ or $\mathcal{I}_s(\psi) = N$

- $\mathcal{I}_s(\forall x\varphi) = T \Leftrightarrow \{\mathbf{d} \in \mathbf{D}_s/(\mathcal{I}_s)^{\mathbf{d}}_x(\varphi) = T\} = \mathbf{D}_s$
  $\mathcal{I}_s(\forall x\varphi) = F \Leftrightarrow \{\mathbf{d} \in \mathbf{D}_s/(\mathcal{I}_s)^{\mathbf{d}}_x(\varphi) = F\} \neq \emptyset$
  $\mathcal{I}_s(\forall x\varphi) = N \Leftrightarrow \{\mathbf{d} \in \mathbf{D}_s/(\mathcal{I}_s)^{\mathbf{d}}_x(\varphi) = N\} \neq \emptyset$ and $\{\mathbf{d} \in \mathbf{D}_s/(\mathcal{I}_s)^{\mathbf{d}}_x(\varphi) = F\} = \emptyset$

- $\mathcal{I}_s(\Box\varphi) = T \Leftrightarrow \{\mathbf{w} \in \mathbf{W} \mid \mathbf{sRw}$ and $\mathcal{I}_{\mathbf{w}}(\varphi) = T\ \} = \mathbf{W}$
  $\mathcal{I}_s(\Box\varphi) = F \Leftrightarrow \{\mathbf{w} \in \mathbf{W} \mid \mathbf{sRw}$ and $\mathcal{I}_{\mathbf{w}}(\varphi) = F\ \} \neq \emptyset$
  $\mathcal{I}_s(\Box\varphi) = N \Leftrightarrow \{\mathbf{w} \in \mathbf{W} \mid \mathbf{sRw}$ and $\mathcal{I}_{\mathbf{w}}(\varphi) = N\ \} \neq \emptyset$ and $\{\mathbf{w} \in \mathbf{W} \mid \mathbf{sRw}$and $\mathcal{I}_{\mathbf{w}}(\varphi) = F\ \} = \emptyset$

- $\mathcal{I}_s(@_n\varphi) = T \Leftrightarrow \mathcal{I}_k(\varphi) = T$ for $k = \mathbf{I}_{nom}(n)$
  $\mathcal{I}_s(@_n\varphi) = F \Leftrightarrow \mathcal{I}_k(\varphi) = F$ for $k = \mathbf{I}_{nom}(n)$
  $\mathcal{I}_s(@_n\varphi) = N \Leftrightarrow \mathcal{I}_k(\varphi) = N$ for $k = \mathbf{I}_{nom}(n)$

- $\mathcal{I}_s(@_w\varphi) = T \Leftrightarrow \mathcal{I}_k(\varphi) = T$ for $k = h(w)$
  $\mathcal{I}_s(@_w\varphi) = F \Leftrightarrow \mathcal{I}_k(\varphi) = F$ for $k = h(w)$
  $\mathcal{I}_s(@_w\varphi) = N \Leftrightarrow \mathcal{I}_k(\varphi) = N$ for $k = h(w)$

- $\mathcal{I}_s(\downarrow w.\varphi) = T \Leftrightarrow (\mathcal{I}_s)^s_w(\varphi) = T$
  $\mathcal{I}_s(\downarrow w.\varphi) = F \Leftrightarrow (\mathcal{I}_s)^s_w(\varphi) = F$
  $\mathcal{I}_s(\downarrow w.\varphi) = N \Leftrightarrow (\mathcal{I}_s)^s_w(\varphi) = N$

The concept of satisfiability will be the usual one: a formula is satisfied in a model at a state if its interpretation is true, otherwise (when it is false or null) the formula is unsatisfied.

$\mathcal{A}, h, s \Vdash \varphi \Leftrightarrow \mathcal{I}_s(\varphi) = T$
$\mathcal{A}, h, s \nVdash \varphi \Leftrightarrow \mathcal{I}_s(\varphi) = F$ or $\mathcal{I}_s(\varphi) = N$

3 A logic to translate $\mathcal{QH}(@, \downarrow)$ into

We have the antecedents of correspondence theory for quantified modal logic, in particular:

1. In [**8**] there is a translation from Quantified Modal Logic into 2-sort predicate logic. There were state-relative domains, no complex terms and classical interpretation.

2. In [**3**]:definition of a Partial Heterogeneous Logic to translate Quantified Modal Logic into. there were state-relative domains, complex terms (but rigid terms) and partial interpretation (the same we have defined here). Details on this PHL can be found in [**4**].

3.1 Partial Heterogeneous Logic

We will look for the suitable logic to translate Quantified Hybrid Logic into. It has to be partial and we know the connectives and quantifiers. It has to have variables in the language to talk about the possible states, we know that heterogeneous (many-sorted) logic allows to have as many kinds of variables as we need. In fact, we only need two kinds of variables: one to talk about individuals of the domains and the other to talk about the states. We will refer to this logic as PHL.
PHL is the Partial Heterogeneous Logic containing the Bočvar's three-valued system of connectives. We will take this PHL as a starting point to characterize $\mathcal{QH}(@, \downarrow)$ syntactically by means of a translation function.

Partial Heterogeneous Language

A *heterogeneous language* $\mathcal{L}$ *for* PHL is a set of formal symbols $ALF(\mathcal{L}) \cup EXPR(\mathcal{L})$, where

- $ALF(\mathcal{L}) = CON \cup FUNC \cup PRED \cup \mathcal{V} \cup \{\forall\} \cup \{\neg, \rightarrow, V, \xi, =\}$.We take as primitive connectives the set $\{\neg, \rightarrow, V, \xi\}$ and as primitive quantifier $\forall$, and we have the usual definitions of $\vee, \wedge, \leftrightarrow$, and $\exists$.
  $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is the set of *variables*, and $\mathcal{V}_i$ the set of *variables of genus* i.

- $EXPR(\mathcal{L}) = TERM(\mathcal{L}) \cup FORM(\mathcal{L})$, the set of the *expressions of the language* $\mathcal{L}$ is defined inductively.

  · $TERM(\mathcal{L})$ is the set of *terms of the language*, and $TERM_i(\mathcal{L})$ the set of terms of the specific genus i. The are defined as usually in a many-sorted logic.

· $FORM(\mathcal{L})$ is the set of *formulas of the language* $\mathcal{L}$. Apart from the atomic formulas we have: $\neg\varphi \mid \varphi \to \psi \mid V\varphi \mid \xi\varphi \mid \forall x\varphi$

Partial Heterogeneous Semantics

A *partial heterogeneous structure* $\mathcal{A} = \langle A_1, A_2, I\rangle$ where for each $i = 1, 2$, A_i is the *universe of genus* i of $\mathcal{A}$, and I is the function to interpret $CON \cup FUNC \cup PRED$

The *assignments* of variables are heterogeneous, i.e., they assign an element of the universe of genus 1 or 2 to a variable of genus 1 or 2, respectively. An *interpretation* is a pair $\mathcal{I} = \langle \mathcal{A}, h\rangle$ where h is a heterogeneous assignment of variables and $\mathcal{A}$ is a heterogeneous structure.

The *interpretation* or *denotation* of an expression ϵ (a term t or a formula φ) by an interpretation $\mathcal{I}$ is the suitable one to our partial logic, bearing in mind that in the interpretation of the quantifier we have to take the corresponding universe (1 or 2) to each sort of variables.

- TERMS:
  $\mathcal{I}(x_i) = h(x_i) \in A_i,\ x_i \in \mathcal{V}_i$
  $\mathcal{I}(c_i) = I(c_i) \in A_i,\ c_i \in CON_i$
  $\mathcal{I}(f\epsilon_1...\epsilon_s) = I(f)(\mathcal{I}(\epsilon_1), ..., \mathcal{I}(\epsilon_s)) \in A_1 \cup A_2,\ f \in FUNC$

- FORMULAS ($TV = \{T,\ F,\ N\}$):
  $\mathcal{I}(r\epsilon_1...\epsilon_n) = I(P)(\mathcal{I}(\epsilon_1), ..., \mathcal{I}(\epsilon_s)) \in TV$
  $\mathcal{I}(\neg\varphi) \in TV$
  $\mathcal{I}(\varphi \to \psi) \in TV$
  $\mathcal{I}(V\varphi) \in TV$
  $\mathcal{I}(\xi\varphi) \in TV$

  $\mathcal{I}(\forall x_i\varphi) = T \ \Leftrightarrow\ \{a_i \in A_i \mid \mathcal{I}_x^{a_i}(\varphi) = T\} = A_i$
  $\mathcal{I}(\forall x\varphi) = F \ \Leftrightarrow\ \{a_i \in A_i \mid \mathcal{I}_x^{a_i}(\varphi) = F\} \neq \emptyset$
  $\mathcal{I}(\forall x\varphi) = N \ \Leftrightarrow \{a_i \in A_i \mid \mathcal{I}_x^{a_i}(\varphi) = N\} \neq \emptyset$ and $\{a_i \in A_i \mid \mathcal{I}_x^{a_i}(\varphi) = F\} = \emptyset$

  $\mathcal{I}(\tau_1 = \tau_2) = T \ \Leftrightarrow\ \mathcal{I}(\tau_1) = \mathcal{I}(\tau_2)$
  $\mathcal{I}(\tau_1 = \tau_2) = F \ \Leftrightarrow\ \mathcal{I}(\tau_1) \neq \mathcal{I}(\tau_2),\ \ \tau_1, \tau_2 \in TERM$, where $\mathcal{I}_{x_i}^{a_i} = \langle \mathcal{A}, \mathcal{H}_{x_i}^{a_i}\rangle$ and $\mathcal{H}_{x_i}^{a_i} = (\mathcal{H} - \langle x_i, \mathcal{H}(x_i)\rangle) \cup \{\langle x_i, a_i\rangle\}$.

3.2 Translation from $\mathcal{QH}(@, \downarrow)$ into PHL

Perhaps we have lingered too much on the presentation of PHL, but this logic has to be used for the translation.

The *translation function* TR from every object of our quantified hybrid logic into PHL is defined as follows:

- A quantified hybrid logic language $FUNC \cup NOM \cup CON \cup PRED \cup \mathcal{VAR} \cup \{\forall, \exists, \neg, \vee, \wedge, \rightarrow, \Box, \Diamond, @, \downarrow, =\}$ is translated into a PHL language $\{f' \mid f \in FUNC\} \cup NOM \cup CON \cup \{P' \mid P \in PRED\} \cup \{R, Q\} \cup \mathcal{VAR} \cup \{\forall, \exists, \neg, \vee, \wedge, \rightarrow, V, \xi, =\}$, where R, Q are symbols of the language, with $I(R) : A_2 \times A_2 \longrightarrow \{T, F\}$, $I(Q) : A_2 \times A_1 \longrightarrow \{T, F\}$ and $I(=) = \{(a, a)/a \in A_1 \cup A_2\}$.
- Regarding the expressions of $\mathcal{QH}(@, \downarrow)$, terms are translated into PHL terms and the translation of the formulas is defined recursively taking into account the following notation and definition:

  Notation: Suppose that we have an enumeration of the set $\mathcal{V}_2$.
  $TR(\varphi)[s]$ means that s is its free variable of genus 2, although we write $TRAN(\varphi)$ if the variable is non relevant.
  $TR(\varphi)[s/w]$ means that s was free variable in $TR(\varphi)$ and all the occurrences of s were replaced by the first variable w in the enumeration of $\mathcal{V}_2$ different from s (and eventually different from all the free variables of genus 2of φ).
  Definition: the new connective $\hookrightarrow$ *restricted implication* has the following tables:

	$\alpha \hookrightarrow \beta$		
$\alpha \backslash \beta$	T	F	N
T	T	F	N
F	T	T	T
N	N	N	N

  - TERMS
    $TR(x) = x$ if $x \in FVAR = \mathcal{V}_1$

$TR(c) = c$ if $c \in CON$
$TR(t) = f' TR(t_1)[w_1/s]...TR(t_n)[w_n/s]s$ if $t = ft_1...t_n$ and s is the first variable of genus 2 in the enumeration of $\mathcal{V}_2$ different from the free variables of genus 2 appearing in $TR(t_1), .., TR(t_n)$, and $TR(t_i)[w_i/s]$ means that we have substitute the free variable w_i of genus 2 of $TR(t_i)$ for the new variable s. Thus, the translated term has only s as a free variable of genus 2.

– FORMULAS
$TR(w)[s] = (s = w)$ if $w \in SVAR$
$TR(n)[s] = (s = n)$ if $n \in NOM$
$TR(P\epsilon_1...\epsilon_n)[s] = \xi(QuTR(\epsilon_1)[w_1/s] \wedge ... \wedge QuTR(\epsilon_n)[w_n/s]) \wedge P' TR(\epsilon_1)[w_1/s]...TR(\epsilon_n)[w_n/s]s$
$TR(t_1 = t_2)[s] = TR(t_1)[s] = TR(t_2)[s]$
$TR(\neg\psi)[s] = \neg TR(\psi)[s]$
$TR(\alpha \to \beta)[s] = TR(\alpha)[v/s] \to TR(\beta)[w/s]$
$TR(\forall x\psi)[s] = \forall x(Qsx \hookrightarrow TR(\psi)[s])$
$TR(\Box\psi)[s] = \forall w(Rsw \hookrightarrow TR(\psi)[s/w])$
$TR(@_n\psi)[s] = TR(\psi)[s/n])$
$TR(@_w\psi)[s] = TR(\psi)[w/s])$
$TR(\downarrow w.\psi)[s] = TR(\psi)[s/w])$

Observe that in the translation of an atomic formula we have added the subformula $\xi(Qs\epsilon_1 \wedge ... \wedge Qs\epsilon_n)$ which guarantees the partial semantics because if the interpretation of a term ϵ_i does not belong to the domain of the state $h(s)$, $Qs\epsilon_1 \wedge ... \wedge Qs\epsilon_n$ will be false, then its difalsification will be null, and the hole conjunction will be null too.
Consider also that in the translation of $\forall x\psi$ we use the restricted implication, the reason is to guarantee the quantification in the domain of the state $h(s)$, if Qsx is false then the restricted implication is true even though $TRAN(\psi)[s]$ is null in the state $h(s)$.
Likewise, to translate the quantification over the states in the quantified hybrid metalanguage into the quantification on variables of genus 2 in the partial heterogeneous language, the restricted implication guarantees that we only quantify over the accessible states from a given state.

- The translation of a quantified hybrid assignment is its extension obtained by adding the values for the variables of genus 2.

- The translation of an interpretation is the PHL interpretation obtained by the translation of its structure and its assignment. I will only consider the global quantified hybrid interpretations here, that is, the ones not depending on the state.

TR preserves truth:

$A, h, \mathbf{w} \Vdash \varphi$ in $\mathcal{QH}(@, \downarrow)$ iff $A, h_s^{\mathbf{w}} \Vdash TR_s(\varphi)$ in PHL, or
$A, h \Vdash @_n\varphi$ in $\mathcal{QH}(@, \downarrow)$ iff $A, h \Vdash TR_s(@_n\varphi)$ in PHL
What is the range of TR? Can we define the inverse translation function HT?

The range of TR:

Given a PHL signature: $\{f'\}_{f \in FUN} \cup C_1 \cup C_2 \cup \{P'\}_{P \in PRED} \cup \{R, Q\} \cup V_1 \cup V_2$
$f't_1...t_ks$ with t_i a term of type 1 and $s \in C_2 \cup V_2$
$P't_1...t_ks$ with t_i a term of type 1 and $s \in C_2 \cup V_2$

The range of TR is the fragment generated by the formulas:

$$s = s\prime \mid t = t \mid Rss' \mid Qxs \mid \xi(Qst_1 \wedge ... \wedge Qst_k) \mid P't_1...t_ks$$
$$\xi(Qst_i \wedge Qst_j) \wedge (t_i = t_j) \mid \xi(Qst_1 \wedge ... \wedge Qst_k) \wedge P't_1...t_ks$$
$$\forall x(Qsx \rightarrow \varphi) \mid \forall w(Rsw \rightarrow \varphi) \mid \neg\varphi \mid \varphi \wedge \psi$$

Inverse translation function

We note HT the inverse function of TR.
$HT(s = s\prime) = @_s s'$
$HT(t = t) = (HT(t) = HT(t))$
$HT(Rss') = @_s \Diamond s'$
$HT(Qxs) = @_s \exists y(x = y)$
$HT(\xi(Qst_1 \wedge ... \wedge Qst_k)) = @_s \exists y_1 ... \exists y_k (t_1 = y_1 \wedge ... \wedge t_k = y_k)$
$HT(P't_1...t_ks) = @_s P(HT(t_1), \ldots, HT(t_k))$
$HT(\xi(Qst_i \wedge Qst_j) \wedge (t_i = t_j)) = @_s(HT(t_i) = HT(t_j))$
$HT(\xi(Qst_1 \wedge ... \wedge Qst_k) \wedge P't_1...t_ks) = @_s P(HT(t_1), \ldots, HT(t_k))$
$HT(\forall x(Qsx \rightarrow \varphi)) = @_s \forall x HT(\varphi)$
$HT(\forall w(Rsw \rightarrow \varphi)) = @_s \Box HT(\varphi)$

HT preserves truth:

$A, h \Vdash \varphi$ in PHL iff $A, h \Vdash HT(\varphi)$ in $\mathcal{QH}(@, \downarrow)$

4 Some historically trouble things in quantified modal logic

Now we have to revise three really trouble properties in Quantified Modal Logic. After using translation and inverse translation we are going to prove that they are not generally valid in $\mathcal{QH}(@,\downarrow)$.

Nested domains condition

For all $s \in W$ *then for every accessible world* Rsw, $D_s \subseteq D_w$
This condition is used by some authors in order to avoid technical problems **[5].**
It is writable in PHL: $\forall s \forall x(Qsx \to \forall w(Rsw \to Qwx))$ [ND]
Translation into $\mathcal{QH}(@,\downarrow)$ by using HT : $@_s \forall x(\Box \exists y(x = y)$ [HT(ND)]
HT(ND) is not a valid formula in $\mathcal{QH}(@,\downarrow)$

Barcan Formula

$@_s(\forall x \Box \varphi \to \Box \forall x \varphi)$ [BF]
There are many objections against the validity of the Barcan Formulas [**5,7,8**].
When we have constant domain condition this formula is valid.
Translation into PHL with TR:
$\forall x(Qsx \to \forall w(Rsw \to \varphi)) \to \forall w(Rsw \to \forall x(Qwx \to \varphi))$
It means that "If at a state **s** everything necessarily possesses a certain property φ, then it is necessarily the case that everything (not only at **s**) possesses that property".
In general BF is not a valid formula in $\mathcal{QH}(@,\downarrow)$

Some equality formulas

$@_s((x = y) \to \Box(x = y))$ [LI]
$@_s(\neg(x = y) \to \Box\neg(x = y))$ [LNI]
In [**5**] (with equality), even in the case of constant domain, both formulas were true.
Translation into PHL with TR:
$\xi(Qsx \wedge Qsy) \wedge (x = y) \to \forall w(Rsw \to (\xi(Qwx \wedge Qwy) \wedge (x = y)))$ [TR(LI)]
$\xi(Qsx \wedge Qsy) \wedge (\neg x = y) \to \forall w(Rsw \to (\xi(Qwx \wedge Qwy) \wedge \neg(x = y)))$ [TR(LNI)]

The nested domains condition implies the validity of LI and LNI
LI and LNI are not a valid in $\mathcal{QH}(@,\downarrow)$

5 Conclusions

The more expressive language of $\mathcal{QH}(@, \downarrow)$ allows the coexistence of state-relative domain, non rigid terms, equality and quantification over individuals. Conflictive quantified modal formulas and conditions are not valid anymore. $\mathcal{QH}(@, \downarrow)$ with state-relative domains has to be a partial logic and we can dealing with the undefined formulas but still have a definition of satisfiability in a classical way:

6 References

[1] Areces, C. and de Rijke, M. (2001): "From Description to Hybrid Logics, and Back". In *Advances in Modal Logic*, volume 3. CSLI Publications.

[2] Blackburn,P and Marx, M (2001): "A Tableaux System for Quantified Hybrid Logic". *Proceedings of Methods for Modalities* 2, Amsterdam.

[3] Huertas, A. (1994): *Modal logic and Non-classical logic.* PhD Thesis, University of Barcelona.

[4] Huertas, A. (2004): "Partial heterogeneous logic for modal logic “. In *SUMMA LOGICAE, http://logicae.usal.es/*

[5] Hughes, G.E. and Cresswell, M.J (1968): *An introduction to modal logic.* Methuen.

[6] Parks, Z. (1976): "Investigations into Quantified Modal Logic", *Studia Logica*, 35, 109-125.

[7] Prior, A.N.(1967): *Past, Present and Future*, Clarendon Press, oxford.

[8] Van Benthem, J. (1983): *Modal logic and classical logic.* Bibliopolis, Napoli.